In this section we record some algebraic facts about vector spaces that will be convenient for understanding the linear objects associated to regular surfaces in 2 and higher dimensions.

1. A VECTOR SPACE AND ITS DUAL SPACE

Let $V$ denote a finite dimensional vector space of dimension $k$. A **linear functional** on $V$ is simply a linear function $f : V \to \mathbb{R}$, and the **dual space** $V^*$ is simply the space of all linear functional on $V$.

**Exercise 1**: Prove that the dual space $V^*$ has the structure of a vector space. That is, given $f, g \in V^*$ and $c \in \mathbb{R}$, show that $f + g, cf \in V^*$.

Given a basis $e = \{e_1, \ldots, e_k\}$ of $V$, the corresponding **dual basis** is the set $e^* = \{e_1^*, \ldots, e_k^*\}$, where $e_k^* : V \to \mathbb{R}$ is the linear functional determined by

$$
e_k^*(e_j) = \begin{cases} 1, & \text{if } k = j \\ 0, & \text{otherwise.} \end{cases}$$

**Exercise 2**: Prove that $e_k^*$ is indeed a linear functional. Prove also that $e^*$ is indeed a basis for $V^*$, so that $\dim V = \dim V^* = k$.

**Exercise 3**: Prove that a vector space $V$ is “naturally” isomorphic to its double dual $V^{**} = (V^*)^*$. That is, show that $v \in V$ can be identified with the linear functional on $V^{**}$ given by $f \mapsto f(v)$.

Let $V$ and $W$ be two vector spaces, and let $\text{Hom}(V,W)$ denote the space of all linear maps $A : V \to W$. Given an $A \in \text{Hom}(V,W)$, there is a natural notion of the **dual map** $A^* \in \text{Hom}(W^*, V^*)$, defined as follows: Given $f \in W^*$, $A^*(f) \in V^*$ is the linear functional given by

$$A^*(f)(v) = f(A(v))$$

**Exercise 4**: Prove that $\text{Hom}(V,W)$ is a real vector space. Prove that if $\dim V = k$ and $\dim W = l$, then $\dim \text{Hom}(V,W) = kl$ (Hint: choose bases $\{v_i\}$ and $\{w_j\}$ of $V$ and $W$ respectively, and let $e_{ij} \in \text{Hom}(V,W)$ be determined by

$$e_{ij}(v_k) = \begin{cases} w_j, & \text{if } k = i \\ 0, & \text{otherwise} \end{cases}$$

Prove that each $e_{ij}$ is indeed an element of $\text{Hom}(V,W)$, and that the set $\{e_{ij}\}$ is a basis.)
Exercise 5: Prove that the dual space to \( \text{Hom}(V,W) \) is \( \text{Hom}(W, V) \). (Hint: for \( B \in \text{Hom}(W,V) \), consider the linear functional \( A \mapsto \text{Trace}_V(BA) \))

Exercise 6: Prove that \( A^*(f) \) is indeed a linear functional on \( V \). Moreover, prove that the map \( A \mapsto A^* \) is a linear isomorphism of \( \text{Hom}(V,W) \) onto \( \text{Hom}(W^*, V^*) \)

2. Tensor products of spaces

Given two vector spaces \( V \), and \( W \), we denote by \( V \otimes W \) the space of bilinear maps \( A : V^* \otimes W^* \rightarrow \mathbb{R} \), and refer to this space as the tensor product of \( V \) and \( W \).

Exercise 7: Prove that \( V \otimes W \) is a vector space. If \( V \) and \( W \) have dimension \( k \) and \( l \), respectively, prove that that \( \dim V \otimes W = kl \). (Hint: The argument should be VERY similar to that of Exercise 3. That is, given bases \( \{v_i\} \) and \( \{w_j\} \) of \( V \) and \( W \), resp., let \( v_i \otimes w_j \) be the bi-linear map determined by

\[
v_i \otimes w_j(v^*_s, w^*_t) = \delta_{is}\delta_{jt}.
\]

Prove that \( v_i \otimes w_j \) is in \( V \otimes W \), and that the set \( \{v^*_i \otimes w^*_j\} \) is a basis. )

Exercise 5 should provide some clarity as to the origins of the notation \( V \otimes W \) for the space of bilinear functionals on \( V^* \times W^* \). Once one fixes a basis for \( V \) and \( W \), any bilinear functional can be written as a linear combination of the basis elements \( v_i \otimes w_j \), so that the space is a sort of “product” of the spaces \( V \) and \( W \). Typically in linear algebra, one constructs the tensor product of vector spaces algebraically and then shows that there is a natural identification of the tensor product with the space of all bilinear functionals. However, we simply take it as a definition, which has the advantage or requiring less algebra, but the disadvantage of slight awkwardness.

By Exerices 3 and 5, we know that \( \text{Hom}(V,W^*) \) and \( V^* \otimes W^* \) are vector spaces of the same dimension. In fact, they are extremely closely related. Consider for example an arbitrary \( l \times k \) real matrix \( A \) with \( ij \)-element \( a_{ij} \). We can consider \( A \) as a map from \( \mathbb{R}^k \) to \( \mathbb{R}^l \), by right-multiplying \( A \) by a column vector \( v \), so that

\[
A(v) := Av = \begin{pmatrix}
a_{11} & \cdots & a_{1k} \\
\vdots & \ddots & \vdots \\
a_{l1} & \cdots & a_{lk}
\end{pmatrix}
\begin{pmatrix}
v_1 \\
\vdots \\
v_k
\end{pmatrix},
\]
and we can consider $A$ as a bilinear map from $\mathbb{R}^k \times \mathbb{R}^l \to \mathbb{R}$ in the usual way by setting

$$A(v, w) := w^T Av = \begin{pmatrix} w_1 & \ldots & w_l \end{pmatrix} \begin{pmatrix} a_{11} & \ldots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \ldots & a_{kj} \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_k \end{pmatrix}.$$ 

Thus, the matrix $A$ is both naturally an element of $\text{Hom}(\mathbb{R}^k, \mathbb{R}^l)$ and $\mathbb{R}^k \otimes \mathbb{R}^l$, so that the two spaces are isomorphic. The following exercise states that a similar identification holds in the general case.

**Exercise 8:** Prove that there is a “natural” isomorphism of vector spaces $\text{Hom}(V, W^*) \to V^* \otimes W^*$. (Hint: given $A \in V^* \otimes W^*$, $v \in V$, let $A_v$ denote the map $A_v(w) := A(v, w)$. The prove that the map $v \mapsto A_v$ is an element of $\text{Hom}(V, W^*)$. Prove that this map is injective. Since $\text{Hom}(V, W^*)$ and $V^* \otimes W^*$ have the same dimension, the map is therefore an ismorphism of vector spaces).

What we mean by “natural” in the statement of Exercise 6 is that the isomorphism is constructed without reference to any coordinate system or bases. For example, given a vector space $V$ and a basis $e$, one gets a vector space isomorphism $A : V \to V^*$ where $A$ is the unique linear map that sends $e_i$ to $e_i^*$ (recall the notion of the dual basis from the previous section). However, we don’t consider this isomorphism natural, since it depends on a choice of basis $e$ and different bases $e$ give rise to different isomorphisms, and so the map $A$ is in this sense somewhat arbitrary. This should convince you that there exist isomorphisms that are natural, and those that are not. Thus, by Exercise 6, we can regard the vector spaces $\text{Hom}(V, W^*)$ and $V^* \otimes W^*$ as the same space without any penalty. Finally, since $V \otimes W$ is a vector space, we can again consider its dual space, which bring us to the final exercise of this section.

**Exercise 9:** Prove that the space dual space of $V \otimes W$ is naturally isomorphic to $V^* \otimes W^*$. (Hint: It’s helpful to use the identification $\text{Hom}(V, W^*) = V^* \otimes W^*$)

3. **The trace of a bilinear map**

An inner product on a vector space $V$ is a bilinear map $g : V \times V \to \mathbb{R}$ (i. e., an element of $V^* \otimes V^*$) such that the following two conditions hold:

**symmetry:** $g(v, w) = g(w, v)$, and

**non-degeneracy:** $g(v, v) \geq 0$ with equality if and only if $v = 0$. 

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**Exercise 10:** Prove that the non-degeneracy condition implies that \( g: V \to V^* \) is an isomorphism.

Exercise 10 allows us to consider the inverse of \( g^{-1} \in \text{Hom}(V^*, V) \). A restatement of Exercise 9 gives that \( \text{Hom}(V^*, V) \) is the dual space to \( \text{Hom}(V, V^*) \). That is, \( g^{-1} \) is a linear functional on \( \text{Hom}(V, V^*) \). We then define the *trace of \( A \) relative to \( g \)* as follows:

\[
\text{Trace}_g(A) = g^{-1}(A).
\]

Here, we are regarding \( A \) as an element of \( \text{Hom}(V, V^*) \) and \( g^{-1} \) a linear functional on \( \text{Hom}(V, V^*) \). Note that this definition is completely extrinsic in the sense that it doesn’t require the use of any coordinate system.

### 4. The trace in coordinates

Let \( V \) be a vector space with inner product \( g \). Let \( \{v_i\} \) be a basis for \( V \) and \( A \in V^* \otimes V^* \) a bilinear map on \( V \). Note that we can think of \( g^{-1} \) as an element of \( V \otimes V \). Let \( \{e_i^* \otimes e_j^*\} \) and \( \{e_i \otimes e_j\} \) denote the corresponding bases of \( V^* \otimes V^* \) and \( V \otimes V \), respectively. We can then write

\[
g = \sum_{ij} g_{ij} v_i^* \otimes v_j^* \\
g^{-1} = \sum_{ij} g^{ij} v_i \otimes v_j \\
A = \sum_{ij} A_{ij} v_i^* \otimes v_j^*
\]

We then get that

\[
g^{-1}(A) = \sum_{ij} g^{ij} v_i \otimes v_j (\sum_{s,t} A_{st} v_s^* \otimes v_t^*)
\]

\[
= \sum_{ij, s,t} g^{ij} A_{st} v_i \otimes v_j (\sum_{s,t} (v_s^* \otimes v_t^*))
\]

\[
= \sum_{i,j,s,t} g^{ij} A_{st} \delta_{is} \delta_{jt}
\]

\[
= \sum_{i,j} g^{ij} A_{ij} = \text{Trace}_g(A)
\]