Solutions to Practice Exam 1

1. Let $\sigma$ be a permutation of $[n]$ with $n$ odd. Show that $\prod_{i=1}^{n}(i-\sigma(i))$ is even.

Solution: The sequence $\sigma(1), \sigma(3), \ldots, \sigma(n)$ consists of $(n+1)/2$ distinct letters from $[n]$, at most $(n-1)/2$ of which can be even. Therefore there exists $k$ such that $\sigma(2k-1)$ is odd, and so $(2k-1) - \sigma(2k-1)$ is even making the given product even. \hfill $\square$

2. How many $n \times n$ matrices with entries 0 or 1 have even row and column sums?

Solution: There are $2^{n-1}$ ways to choose a row with an even number of ones, since this is the number of even element subsets of $[n]$. The first $n-1$ rows may be chosen arbitrarily, in $(2^{n-1})^{n-1} = 2^{(n-1)^2}$ ways. The final row is determined by the condition that column sums must be even. Since all column sums are even, the total number of 1’s is be even, and since the first $n-1$ row sums are all even, this ensures that the final row will also be even. Therefore the total number of matrices is $(2^{n-1})^{n-1} = 2^{(n-1)^2}$. \hfill $\square$

3. Give two proofs that $\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}$, one combinatorial and one using Pascal’s identity.

Solution 1: Both sides count the number of binary strings of length $n+1$ with exactly $r+1$ ones. The left-hand side is straightforward: of the $n+1$ positions, choose $r+1$ of them to have value 1. For the right-hand side, the final 1 in the string must occur at position $k = r+1, r+2, \ldots, n+1$, and there must be $r$ additional 1’s in the first $k-1$ positions. \hfill $\square$

Solution 2: When $n = 0$, the identity reduces to $1 = 1$ if $r = 0$ and $0 = 0$ otherwise. Using Pascal’s identity and induction on $n$, we have

$$\binom{n+1}{r+1} = \binom{n}{r} + \binom{n}{r+1} = \binom{n}{r} + \sum_{j=r}^{n-1} \binom{j}{r} = \sum_{j=r}^{n} \binom{j}{r}.$$ \hfill $\square$

4. Let $f_n$ be the number of compositions of $n$ into parts of sizes 1 and 2. Find the generating function of $\{f_n\}_{n \geq 0}$.

Solution: Removing the first part of a composition of $n$ into parts of sizes 1 and 2 gives such a composition into parts of sizes 1 and 2 of $n-1$ if $\alpha_1 = 1$ and of $n-2$ if $\alpha_2 = 2$. Therefore $f_n$ satisfies the recurrence $f_{n+2} = f_{n+1} + f_n$, with initial conditions $f_0 = 1$ and $f_1 = 1$. Letting $F(x)$ be the corresponding generating function, we have

$$\sum_{n \geq 0} f_{n+2}x^{n+2} = \sum_{n \geq 0} f_{n+1}x^{n+2} + \sum_{n \geq 0} f_{n}x^{n+2} \quad \Rightarrow \quad F(x) - x - 1 = x(F(x) - 1) + x^2F(x)$$

Solving this equation for $F(x)$ gives $F(x) = 1/(1 - x - x^2)$. \hfill $\square$

5. Alice and Bob invite $n$ couples to their house for Thanksgiving dinner, and the couples bring a total of $k$ children. They have two circular tables for adults, each accommodating $n$ people, and a shorter circular table for the kids. In how many ways can Alice, Bob and all of their guests sit down for dinner?

Solution: The adults must sit around 2 circular tables each accommodating $n$ people, which is equivalent to giving a permutation of $[2n]$ with cycle type $(n,n)$. The $k$ children sit around a single table, which is a permutation of $[k]$ with cycle type $(k)$. Therefore the total number of seating arrangements is

$$\binom{\frac{(2n)!}{2n^2}}{\frac{k!}{k}} = \frac{(2n-1)!(k-1)!}{n}.$$ \hfill $\square$