Solutions to Problem Set 1

1. (2 points) The contiguous United States is less than 3000 miles east to west and less than 2000 miles north to south. Therefore we may divide the entire area into six $1000 \times 1000$ mile regions. By the Pigeon-hole Principle, for any 7 people, there must be two living inside the same $1000 \times 1000$ square. By the Pythagorean Theorem, the greatest distance between these two people is $1000\sqrt{2} < 1500$ miles.

2. (3 points) For $i = 1, 2, \ldots, n$, let $A_i = \{2i - 1, 2i\}$. Since $A_1 \cup A_2 \cup \cdots \cup A_n = \{1, 2, \ldots, 2n\}$, by the Pigeon-hole Principle, if we select $n+1$ integers from $\{1, 2, \ldots, 2n\}$, there must be some $i$ so that two come from $A_i$, and these two will have greatest common divisor 1. For the second part, repeat the argument for the sets $B_i = \{2^k(2i-1)\}$.

3. (3 points) If we chose $N$ cards, then by the generalized Pigeon-hole Principle, we must have at least $\lceil N/4 \rceil$ cards of the same suit. The smallest value of $N$ such that $\lceil N/4 \rceil \geq 3$ is $N = 9$. Moreover, if we have only 8 cards, we may take 2 from each suit. The Pigeon-hole Principle cannot be used for the second part of the problem. Since there are 39 non-hearts, we can have up to 41 cards without having three hearts. However, with 42 cards, there must be three hearts.

4. (2 points) Choose one person, say Alice, and consider her relation to the remaining five people. By the generalized Pigeon-hole Principle, either Alice has at least three acquaintances or at least three are strangers to her. In the former case, either all three of Alice’s acquaintances are mutual strangers or else two of these three are acquainted, in which case they, together with Alice, form three mutual acquaintances. In the latter case, either all three strangers to Alice are mutual acquaintances or else two of these are strangers, in which case they, together with Alice, form three mutual strangers.