Problem Set 12

Due 12/03:

1. (2 points) A two-headed coin, a two-tailed coin and an ordinary coin are placed in a bag. One of the coins is drawn at random, flipped, and comes up heads. What is the probability that there is a head on the other side of this coin?

2. (2 points) For some fixed $b, r > 0$, place $b$ blue balls and $r$ red balls in a box. Select balls from the box, one after the other, without replacement. Prove that the probability that the $k$th ball taken from the box is blue is the same for $1 \leq k \leq b$.

3. (2 points) If, after grading an exam for a class of $n$ students, I randomly hand back the exams to the class so that each person gets one exam, what is the probability that no one gets his or her own exam back?

4. (2 points) Alice is taking a quiz consisting of one multiple choice question with three possible solutions, exactly one of which is correct. She has absolutely no idea which the correct answer is, so she randomly circles one of the choices. Afterwards, she see that Bob has circled a different choice, and she knows for certain that Bob always answers every question incorrectly. Alice then decides to change her answer to the option that neither she nor Bob has circled. Does this increase, decrease or leave unchanged the probability that Alice correctly guesses the answer?

5. (2 points) Prove that the number of length $2n$ binary strings $s$ with exactly $n$ 1’s such that for all $i$ the sub-string $s_1s_2 \cdots s_i$ has at least as many 1’s as 0’s is $\frac{1}{n+1} \binom{2n}{n}$.

6. (2 points) Prove that the number of triangulations of a regular $n+2$-gon is $\frac{1}{n+1} \binom{2n}{n}$.

7. (2 points) Prove that the number of $2 \times n$ arrays of the numbers $1, 2, \ldots, 2n$ with increasing rows and columns, i.e. $\begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{pmatrix}$ with $a_1 < a_2 < \cdots < a_n$ and $b_1 < b_2 < \cdots < b_n$ and $b_i < a_i$ for $i = 1, 2, \ldots, n$, is $\frac{1}{n+1} \binom{2n}{n}$.

8. (2 points) Prove that the number of words $w$ of length $n$ on the alphabet $[n]$ such that $w_{n-i} > i$ and $w_i \leq w_{i+1} + 1$ is $\frac{1}{n+1} \binom{2n}{n}$.