Problem Set 8

Practice: 10.9, 10.10, 10.13, 10.17

Due 11/03:

1. (2 points) The complement of a graph $G$, denoted $\overline{G}$, is the graph on the same vertex set as $G$ with an edge between $u$ and $v$ if and only if $G$ does not have an edge between $u$ and $v$. Find the smallest constant $N$ such that for all graphs $G$ on $n$ vertices with $n > N$, at least one of $G$ and $\overline{G}$ contains a cycle.

2. (2 points) Corollary 10.18 in the text is false. Explain the flaw in the proof and give a counter-example to the statement.

3. (2 points) Prove that the number of trees on $[n]$ with $\deg(i) = d_i$ is given by

$$\binom{n-2}{d_1-1,d_2-1,\ldots,d_n-1}$$

4. (2 points) Show that the number of spanning trees of any loopless graph $G$ on $[n]$ is at least $\sqrt{(d_1-1)\cdots(d_n-1)}$, where $d_i$ is the degree of vertex $i$.

5. (2 points) Let $C_n$ be the cycle on $[n]$ with an edge between $i$ and $j$ if and only if $i - j = \pm 1 \mod n$. Compute the number of spanning forests of $C_n$ with $k$ trees.

6. (2 points) The $n$-dimensional hypercube, denoted $Q_n$, is the simple graph with vertex set $(v_1,v_2,\ldots,v_n) \in \{0,1\}^n \subset \mathbb{R}^n$ and edges between vertices which differ in exactly one coordinate. Compute the number of spanning trees of $Q_n$.

7. (2 points) Count the number of Eulerian walks on the de Bruijn graph.

8. (2 points) Prove that the following determinant is a positive integer whenever $k \leq n$

$$\begin{vmatrix} \binom{n}{k} & \binom{n}{k+1} & \binom{n}{k+2} \\ \binom{n}{k-1} & \binom{n}{k} & \binom{n}{k+1} \\ \binom{n}{k-2} & \binom{n}{k-1} & \binom{n}{k} \end{vmatrix}$$