18.02 Problem Set 7 (due Friday April 2, 1:45PM, 2-106)

Part I:

Hand in the underlined problems; the others are for practice.


Special Problem: The original problem was to evaluate
\[ \iint_R (2x - 3y)^2 (x + y)^2 \, dx\, dy \]
over \( R \): where \( R \) is the region to the right of the \( y \)-axis, below the \( x \)-axis and above the line \( 2x - 3y = 4 \). However this integral is undefined as a Riemann integral (because the integrand is unbounded) or, more sensibly is \(+\infty\) as a Lebesgue integral (or as an improper Riemann integral). In short I should have been more careful when I copied this problem, it is bad, bad bad! The difficult is that the line \( x + y = 0 \) where the integrand is infinite is inside the domain \( R \).

If instead I had asked you to evaluate
\[ \iint_R (2x - 3y)^2 (x + y + 3)^2 \, dx\, dy \]
for the same \( R \) we would have had less trouble!

Solutions: EP page 955, no. 9: For \( u = xy \), \( v = xy^3 \) the Jacobian is
\[ \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} y & x \\ y^3 & 3xy^2 \end{vmatrix} = 2xy^3. \]
Thus \( \frac{\partial(x,y)}{\partial(u,v)} = 1/2v \). The area integral is therefore
\[ \int_0^6 \int_2^4 \frac{du\, dv}{2v} = \int_0^6 \frac{dv}{2v} = \ln 2. \]

EP page 956, no. 13: Since \( x = 3r \cos \theta \) and \( y = 2r \sin \theta \), the Jacobian is
\[ \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} 3 \cos \theta & -3r \sin \theta \\ 2 \sin \theta & 2r \cos \theta \end{vmatrix} = 6r. \]
The integral for the volume is
\[ \int \int_R (x^2 + y^2)^6 r\, dr\, d\theta = \int_0^{2\pi} \int_0^1 6r^3(4 + 5 \cos^2 \theta) \, dr\, d\theta \]
\[ = \int_0^{2\pi} \frac{3}{2} (4 + 5 \cos^2 \theta) \, d\theta = \frac{39\pi}{2}. \]

Special Problem: Make the change of variables \( v = 2x - 3y \), \( u = x + y \). The Jacobian is
\[ \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = -5. \]
In terms of \( u \) and \( v \) the region of integration is between \( v = 0 \) and \( v = 4 \) and when \( v = 2x - 3y \) is fixed, \( u \) runs from \( x = 0 \), where it is \( y = -v/3 \), to \( y = 0 \) where it is \( x = v/2 \). I leave it as an exercise to evaluate the resulting iterated integral
\[ \int_0^4 \int_{-v/3}^{v/2} \frac{u^2}{(u + 3)^2} \frac{du\, dv}{5}. \]


Lecture 21. Tues. (March 29): Path independence; conservative fields in the plane. Read 15.3 to p. 979. Problem: Work SN Vect. Calc. page 2.7 1, 2, 3 (S.27) However, the solution to Problem 3 on page S.27 is incorrect. Rather, \( F = \nabla f = \cos x \cos y \mathbf{i} - \sin x \sin y \mathbf{j} \). The rest is correct.

Remark: The Fundamental Theorem of Calculus for line integrals is Theorem 1. You should be able to state and prove the theorem (in the plane; ignore \( z \)). The book writes \( \int_C F \cdot \mathbf{T} ds \), in the lectures and notes we use \( \int_C F \cdot \mathbf{dr} \). Both have the same meaning: the line integral which calculates the work done by the field \( F \) carrying a unit test object along the curve \( C \).

Part II: (15 pt)

Directions: Try each problem alone for 15 minutes. If you subsequently collaborate, this should be acknowledged and solutions must be written up independently.


Solution: The integral is 

\[
I = \int \int_R \exp \left( \frac{x-y}{x+y} \right) dx \, dy.
\]

For the suggested change of variables, \( u = x - y, v = x + y \) the Jacobian is

\[
\begin{vmatrix}
\frac{\partial(u,v)}{\partial(x,y)} & \\
1 & 1 \\
1 & 1
\end{vmatrix} = 2.
\]

Thus \( dx \, dy = \frac{1}{2} du \, dv \). The region of integration, \( R \), is the set where \( x, y \geq 0 \) and \( x + y \leq 1 \). In terms of the new variables this is \( 0 \leq v \leq 1, u + v \geq 0, u - v \geq 0 \).

Thus

\[
I = \frac{1}{2} \int_0^1 \int_{-v}^v \exp(u/v) du \, dv = \frac{1}{2} \int_0^1 \left[ v \exp(u/v) \right]_{-v}^v = \frac{1}{2} \int_0^1 v(e - \frac{1}{e}) = \frac{1}{4}(e - \frac{1}{e}).
\]

Problem 2. (Fri. 2 pt) Write down the velocity field for a standard 2-dimensional flow between the lines \( x = 0 \) and \( x = 1 \): The flow is upwards, with parabolic cross-section; i.e., along any horizontal line segment between 0 and 1, the velocity vector has magnitude 0 at the two ends, while in between its length increases and decreases so the tips of the vectors lie on a parabola, whose maximum height is 1, in the middle. Indicate reasoning. (This is the way a liquid flows in a pipe if it adheres to the pipe walls.)

Solution: The direction of the vector field is always \( \mathbf{j} \). Its magnitude must be \( |\mathbf{F}| = 4x(1-x) \), since it has to be quadratic and vanish at \( x = 0 \) and \( x = 1 \) and have size 1 when \( x = \frac{1}{2} \). Thus

\[
\mathbf{F} = 4x(1-x)\mathbf{j}.
\]

Problem 3. (Fri. 3 pt) Imagine the \( z \)-axis represents an infinitely long, uniformly charged wire. The electric force it exerts on a unit charge at the point \((x, y)\) is given by

\[
F(x, y) = k(x\mathbf{i} + y\mathbf{j})/(x^2 + y^2).
\]

Find by direct calculation of the line integral in each case the work done by the force in moving a unit charge along the paths:
1. line from \((0, 1)\) to \((\infty, 1)\).
2. circle of radius \(a\) with center at origin, traced counterclockwise.
3. line from \((1, 0)\) to \((0, 1)\) (use integral tables in the book covers).

Solution:
The vector field is 
\[
F = \frac{kxi + kyj}{x^2 + y^2}
\]
so the work done along a curve \(C\) is
\[
k \int_C \frac{xdx + ydy}{x^2 + y^2}.
\]

1. The curve is \(x\) running from 0 to \(\infty\) while \(y = 1\). Thus the work done is
\[
k \int_0^\infty \frac{xdx}{x^2 + 1} = \frac{k}{2} \ln(x^2 + 1)|_0^\infty = \infty.
\]
So an infinite amount of work must be done (just like 18.02!)

2. The curve is \(x = a\cos t, y = a\sin t, t\) running from 0 to \(\pi/2\). Thus \(dx = -a\sin t\, dt, dy = a\cos t\, dt\) and the work done is
\[
k \int_0^{\pi/4} -a^2 \cos t\sin t + a^2 \sin t\cos t \frac{dt}{a^2} = 0.
\]

3. Parameterize by \(y = 1 - x, dy = -dx, x\) running from 1 to 0, so the work done is
\[
k \int_1^0 \frac{xdx + (1 - x)(-dx)}{x^2 + (1 - x)^2} = \int_1^0 \frac{(2x - 1)dx}{2x^2 - 2x + 1} = \frac{1}{2} \ln(2x^2 - 2x + 1)|_1^0 = 0.
\]

**Problem 4.** (Fri. 8 pt) Answer the same questions as in SN Vect. Calc. page 2.7 no. 1 for the function \(f(x, y) = xy(x + y)\), and the path \(C\) given by the quarter circle running from \((0, 1)\) to \((-1, 0)\).

Solution:
a) Since \(f = x^2 y + xy^2, F = (2xy + y^2)i + (x^2 + 2xy)j\).
b) Directly: Parameterize the curve by \(x = \cos t, y = \sin t, \pi/2 \leq t \leq \pi\). Thus \(dx = -\sin t\, dt, dy = \cos t\, dt\) so the line integral is
\[
\int_{\pi/2}^\pi (-\sin^3 t - 2 \cos t \sin^2 t + \cos^3 t + 2 \sin t \cos^2 t)\, dt.
\]
Using the formule \(\sin^3 t = \sin t(1 - \cos^2 t)\) and \(\cos^3 t = \cos t(1 - \sin^2 t)\) the integral becomes
\[
\int_{\pi/2}^\pi (-\sin t - 3 \cos t \sin^2 t + \cos t + 3 \sin t \cos^2 t)\, dt = \cos t - \sin^3 t - \sin t - \cos^3 t|_{\pi/2}^\pi = 0.
\]
A simpler path between these two point is down the y-axis to the origin and then along the x-axis. The integrand vanishes on both pieces so the integral is zero. By the fundamental theorem
\[
\int_C F \cdot dr = f(x, y)|_{(0, 1)}^{(1, 0)} = 0.
\]