1. Compute the volume of the region between the $xy$ plane and the surface $z = -x^2 - y^2 + 4$.

Solution: The volume is given by the double integral over the region, $R = \{x^2 + y^2 \leq 4\}$

$$\int \int_R (4 - x^2 - y^2)\,dA = \int_0^{2\pi} \int_0^2 (4 - r^2) r\,d\theta\,dr$$

$$= 2\pi \left[ 2r^2 - \frac{r^4}{4} \right]_0^2 = 8\pi.$$

2. Evaluate the integral

$$\int_0^1 \int_x^1 \cos(y^2)\,dy\,dx.$$

Solution: Change the order of integration and the integral becomes

$$\int_0^1 \int_0^y \cos(y^2)\,dx\,dy = \int_0^1 y\cos(y^2)\,dy = \left[ \frac{1}{2} \sin(y^2) \right]_0^1 = \frac{1}{2} \sin 1.$$

3. Using polar coordinates evaluate the integral

$$\int \int_R \exp(x^2 + y^2)\,dA$$

where $R$ is the disc of radius 1 and center the origin.

Solution: In polar coordinates the integral is

$$\int_0^{2\pi} \int_0^1 \exp(r^2) r\,dr\,d\theta = 2\pi \left[ \frac{1}{2} \exp(r^2) \right]_0^1 = \pi(e - 1).$$
4. (a) For what values of the constant $a$ is
\[ \mathbf{F} = (4x^3 + 2xy + ay^2)i + (x^2 + 2y)j \]
a conservative vector field?
(b) For $a = 0$ compute
\[ \int_C \mathbf{F} \cdot d\mathbf{r} \]
where $C$ is the semicircle with radius 1, center $(0, 0)$ from $(1, 0)$ to $(-1, 0)$ in the upper half-plane.

Solution:
(a) \[ \frac{\partial}{\partial y} (4x^3 + 2xy + ay^2) = 2x + 2ay = \frac{\partial}{\partial x} (x^2 + 2y) = 2x \]
exactly when $a = 0$.
(b) Since the vector field is conservative the integral is the same for any contour with these two endpoints, for instance the segment $[-1, 1]$ of the $x$-axis. The integral along the $x$-axis is \[ \int_{-1}^{1} 4x^3 dx = 0. \]

5. (a) For the vector field $\mathbf{F} = y(x^2 + y^2)^2i - x(x^2 + y^2)^2j$ compute $\text{div}(\mathbf{F})$
(b) For this vector field and the curve $C$ which is the circle of radius 2 with center the origin and positive orientation, use Green’s theorem to compute the flux integral
\[ \oint_C \mathbf{F} \cdot \hat{n} \, ds. \]

Solution:
(a) \[ \text{div}(\mathbf{F}) = \frac{\partial}{\partial x} [y(x^2 + y^2)^2] - \frac{\partial}{\partial y} [x(x^2 + y^2)^2] \]
\[ = 4xy(x^2 + y^2) - 4xy(x^2 + y^2) = 0. \]
(b) 0
6. Sketch the parameterized curve  
\[ x = \cos t, \quad y = \sin^3 t, \quad 0 \leq t \leq 2\pi. \]

(a) Why is it closed?
(b) Why is it simple?
(c) Use Green’s theorem to express the area inside the curve as a single integral. Do not evaluate it.

OR simply:
Use Green’s theorem to express the area inside the simple closed curve
\[ x = \cos t, \quad y = \sin^3 t, \quad 0 \leq t \leq 2\pi, \]
as a single integral. Do not evaluate it.

Solution: [There is of course a small conceptual problem here that the curve is not smooth, but I don’t suppose that anyone will notice.]
(a) \( x(2\pi) = x(0), \quad y(2\pi) = y(0). \)
(b) \( \sin t \) takes each value between \(-1\) and \(1\) twice for \( t \) in the interval \([0, 2\pi]\) but at these points \( \sin t \), and hence \( \sin^3 t \) has opposite sign.

(c)
\[ \oint_C -y \, dx = \int_0^{2\pi} \sin^4 t \, dt. \]

7. Let \( R \) be the region in the first quadrant \( (x \geq 0, y \geq 0) \) where \( 3x + y \leq 1 \). Using the change of variable formula for double integrals rewrite the integral
\[ \int_R \int (3x + y)^4 \exp(x + y) \, dA \]
as an iterated integral in terms of the new variables \( u = 3x + y \) and \( v = x + y \). Give limits of integration but do not evaluate.

Solution: For \( u = 3x + y \) and \( v = x + y \) the Jacobian is
\[ \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} = 2, \]
so \( \partial(x,y)/\partial(u,v) = 1/2 \). On \( x = 0, \quad v = u \) and on \( y = 0, \quad v = u/3 \) so the integral becomes
\[ \frac{1}{2} \int_0^1 \int_{u/3}^u u^4 \exp(v) \, dv \, du. \]