Problem 1 (15 points) The solid $R$ is the piece of the first octant cut off by the plane
\[ x + y + z = 1 \]
Set up an iterated double integral in rectangular coordinates which gives the volume of $R$. (Give the integrand and limits of integration, but do not evaluate.)

\[ \int \int_R \, dA \]

Problem 2 (15 points) A circular disc has radius $a$ and $A$ is a point on its circumference. The density at any point $P$ on the disc is equal to the distance of $P$ from $A$. Set up an iterated double integral in polar coordinates which gives the mass of the disc. Place $A$ at the origin. (Give the integrand and limits, but to not evaluate the integral.)

Problem 3 (15 points) Evaluate the integral
\[ \int_0^1 \int_0^1 \cos(y^2) \, dy \, dx \]
by changing the order of integration. (Sketch the region of integration first.)

Problem 4 (10 points) Change the double integral
\[ \int_0^1 \int_{x^2}^{x} \frac{(x^2 + y^2)^{3/2}}{2} \, dy \, dx \]
to an iterated integral in polar coordinates. (Do not evaluate it.)

Problem 5 (30 points; 10 each)

a) $F = (axy + y^2)i + (x^2 + bxy + 1)j$; $a, b$ are constants. Show that $F$ is conservative $\iff a = 2, b = 2.$

b) Taking $a = 2, b = 2$, find $f(x, y)$ so that $F = \nabla f$.

c) Still taking $a = 2, b = 2$, show $\int_C F \cdot dr = 0$ for any curve $C$ beginning and ending on the x-axis.

Problem 6 (30 points; 15, 5, 10)

a) Evaluate $\int_C -x^2 y \, dx + xy^2 \, dy$ by Green’s theorem, if $C$ is the closed curve as pictured passing through $(1, 0), (\sqrt{2}, \sqrt{2}), (0, 0)$, and back to $(1, 0)$.
b) Show that for any simple closed curve $C$ directed positively,
\[ \oint_C -y \, dx = \text{Area inside } C. \]
c) The curve $y^2 = x^2(1 - x)$ shown is given parametrically by
\[ x = 1 - t^2, \quad y = t - t^3. \]
Find the area inside the loop.

Problem 7 (15 points; 5, 10)
a) Write down in rectangular coordinates the field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ whose vectors all have unit length and point radially outward from the origin. ($\mathbf{F} = 0$ at (0, 0).)
b) For this field, give the values of $\int_C \mathbf{F} \cdot d\mathbf{r}$ over the following curves (no calculation is required):
\begin{itemize}
  \item $C_1$ is the unit semi-circle in the upper half-plane, running from (1, 0) to (-1, 0)
  \item $C_2$ is the line segment from (0, 0) to (1, 1)
\end{itemize}
Brief solutions.

Problem 1

Thus the volume is $\int_0^1 \int_0^{1-x} (1 - x - y) \, dx \, dy$ OR $\int_0^1 \int_0^{1-x} (1 - x - y) \, dx \, dy$.

Problem 2

With the center on the x-axis
\[ \text{Mass} = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos \theta} r \cdot r \, dr \, d\theta \]
OR with the center on the y-axis:
\[ \text{Mass} = \int_0^\pi \int_0^{2\sin \theta} r \cdot r \, dr \, d\theta \]

Problem 3 The region of integration for $\int_0^1 \int_x^1$ is
so the integral becomes \( \int_0^1 \int_0^y \cos(y^2) \, dx \, dy \), which evaluates by

**Inner:** \( \cos(y^2) \cdot x \big|_0^1 = \cos(y^2) \cdot y \)

**Outer:** \( \frac{1}{2} \sin(y^2) \big|_0^1 = \frac{1}{2} \sin 1 \)

**Problem 4** The region of integration for \( \int_0^1 \int_{-x}^x \ldots \, dy \, dx \) is \( (r^2 = x^2 + y^2) \)

so in polar coordinates the integral becomes

\[ \int_{-\pi/4}^{\pi/4} \int_0^{\sec \theta} r^3 \times r \, dr \, d\theta. \]

**Problem 5**

1. \( \vec{F} \) conservative \( \iff \frac{\partial(axy+y^2)}{\partial y} = \frac{\partial(x^2+4xyy+1)}{\partial x} \iff ax + 2y = 2x + by \iff \)

\[ a = 2, \quad b = 2 \]

2. **Method 1:**

\[
\begin{align*}
&\begin{align*}
&\frac{1}{2} \int_1^{(x_1, y_1)} \frac{1}{2} \int_0^{(x_1, 0)} f(x, y) = f_1 + f_2 \quad \int_1 = 0 \text{ since along } 1 \, y = 0 \, dy = 0 \\
&f_2 = \int_0^{x_1^2 + 2x_1y + 1} dy = x_1^2 y_1 + x_1 y_1^2 + y_1 \\
&\text{since } x < x_1, \, dx = 0 \text{ along path 2}
\end{align*}
\end{align*}
\]

OR

3. **Method 2:** \( \frac{\partial f}{\partial x} = 2xy + y^2 \)

\[
\begin{align*}
\therefore f &= x^2 y + xy^2 + g(y) \\
\frac{\partial f}{\partial y} &= x^2 + 2xy + g'(y) = x^2 + 2xy + 1 \\
g'(y) &= 1, \quad g(y) = y
\end{align*}
\]
4. Using fundamental theorems:
\[ \int_{(x_0,0)}^{(x_1,0)} \vec{F} \cdot d\vec{r} = \int_{(x_1,0)}^{(x_0,0)} \nabla(x^2y + xy^2 + y) \cdot d\vec{r} = 0 - 0 = 0 \]

OR

Since \( \vec{F} \) is path-independent, we can replace \( C \) by a path \( D \) on the \( x \)-axis:
\[ \int_C \vec{F} \cdot d\vec{r} = \int_D \vec{F} \cdot d\vec{r} = \int_{x_0}^{x_1} 0 \text{ (since } y = 0 \text{ on } D) \cdot dx = 0. \]

Problem 6
1. \( \oint_C -x^2y \, dx + xy^2 \, dy = \iint_R y^2 - (-x^2) \, dA \) by Green’s theorem

\[\begin{align*}
\iint_R y^2 - (-x^2) \, dA &= \int_0^{\pi/4} \int_0^1 r^2 \, r \, dr \, d\theta = \frac{\pi}{4} \left[ \frac{r^4}{4} \right]^1_0 = \frac{\pi}{16}.
\end{align*}\]

The original picture can be interpreted to mean that \( C \) was not closed, with the part in the \( x \)-axis missing. Since this actually contributes nothing to the line integral, the answer is the same!

2. By Green’s theorem, \( \oint_C -y \, dx = \iint_R \frac{\partial(-y)}{\partial y} \, dA = \iint_R dA = \text{area of } R \)

3. Using (b), area inside loop
\[\begin{align*}
\oint_C -y \, dx &= \int_{-1}^1 -t + t^3(-2t) \, dt = \int_{-1}^1 (2t^2 - 2t^4) \, dt \\
&= \frac{2t^3}{3} \bigg|_{-1}^1 = \frac{2}{3} - \frac{2}{3} - \left( \frac{-2}{3} + \frac{2}{3} \right) = \frac{8}{15}
\end{align*}\]

Problem 7
a) \( \vec{F} = \frac{x\hat{i} + y\hat{j}}{\sqrt{x^2 + y^2}} \) has magnitude = 1 (since \( |x\hat{i} + y\hat{j}| = \sqrt{x^2 + y^2} \)) and has radially outward direction.
b) For $C_1$, the unit semi-circle in the upper half-plane, running from $(1,0)$ to $(-1,0)$, $\mathbf{F}$ is perpendicular to the tangent of the curve, at every point, so the integral is zero.

For $C_2$, $\mathbf{F}$ and $C$ have the same direction and both are constant (on $C_2$), so the work is simply $|\mathbf{F}| \times \text{(distance)} = 1 \cdot \sqrt{2} = \sqrt{2}$.