Topological Hochschild homology and the de Rham-Witt complex for $\mathbb{Z}(p)$-algebras

Lars Hesselholt

Abstract. This paper shows that for a $\mathbb{Z}(p)$-algebra ($p$ odd), the equivariant homotopy groups in degrees less than or equal to one of the topological Hochschild $T$-spectrum are given, as functors, by the de Rham-Witt complex.

Introduction

Let $A$ be a commutative and unital $\mathbb{Z}(p)$-algebra with $p$ odd. The topological Hochschild spectrum $TH(A)$ has a natural action by the circle $\mathbb{T}$, and one defines

$$\text{TR}_n(A; p) = TH(A)^{C_{p^n-1}}$$

as the fixed point spectrum for the cyclic subgroup of the indicated order. Typically, the homotopy groups

$$\text{TR}_n(A; p) = \pi_\ast \text{TR}^1(A; p)$$

are very large, but they have a rich algebraic structure. There is a universal example of this structure, the de Rham-Witt complex, and the canonical map

$$\lambda: W_n \Omega^n \rightarrow \text{TR}_n(A; p)$$

is in many ways analogous to the map from Milnor $K$-theory to Quillen $K$-theory. For example, it is an isomorphism of pro-abelian groups if $A$ is a regular $F_p$-algebra by [4, theorem B]. In this paper we prove the following result whose $K$-theory analog is well-known.

Theorem. Let $A$ be a $\mathbb{Z}(p)$-algebra with $p$ odd. Then the canonical map

$$\lambda: W_n \Omega^n \rightarrow \text{TR}_n(A; p)$$

is an isomorphism, for $q \leq 1$.

The statement was known previously in a number of special cases. The case of a polynomial algebra over $\mathbb{Z}(p)$, proved in [5], is the starting point of the argument given here. We also mention that in his thesis [9], Kåre Nielsen has verified the case of a truncated polynomial algebra in a finite number of variables over a perfect field of characteristic $p > 0$. He shows further that if $A = F_p[x]/(x^q)$ and $q = 2$,
the map $\lambda$ is not pro-isomorphism. However, it seems reasonable to expect that $\lambda$ is a pro-isomorphism for $q = 2$, if $A$ is regular.

In this paper, a pro-object in a category $C$ is a functor from the set of positive integers, viewed as a category with one arrow from $n + 1$ to $n$, to $C$, and a strict map between pro-objects is a natural transformation.

1. The de Rham-Witt complex

1.1. We briefly recall the definition of the de Rham-Witt complex and refer to [5] for details. This definition extends to $\mathbb{Z}(p)$-algebras (with $p$ odd) the original definition for $\mathbb{F}_p$-algebras of Bloch-Deligne-Illusie [8].

Let $A$ be a $\mathbb{Z}(p)$-algebra (with $p$ odd). By a Witt complex over $A$, we mean the following data:

(i) a pro-differential graded ring $E^*_n$ and a strict map of pro-rings

$$\lambda: W(A) \to E^0$$

from the pro-ring of Witt vectors in $A$;

(ii) a strict map of pro-graded rings

$$F: E^*_n \to E^*_{n-1},$$

such that $\lambda F = F\lambda$ and such that for all $a \in A$,

$$Fd\lambda[a]_n = \lambda[a]_n^{p-1}d\lambda[a]_{n-1},$$

where $[a]_n = (a, 0, \ldots, 0) \in W_n(A)$ is the multiplicative representative;

(iii) a strict map of pro-graded $E^*_n$-modules

$$V: F, E^*_{n-1} \to E^*_n,$$

such that $\lambda V = V\lambda$ and such that $FV = p$ and $FdV = d$. (Here $F, E^*_{n-1}$ denotes the $E^*_{n-1}$-module $E^*_n$ considered as an $E^*_n$-module via $F: E^*_n \to E^*_{n-1}$.)

By a map of Witt complexes we mean a strict map of pro-differential graded rings $f: E^*_n \to E'^*_{n}$ such that $f\lambda = \lambda' f$, $fF = F' f$, and $fV = V' f$. We write $R$ for the structure map in the pro-graded ring $E^*_n$ and call it the restriction map.

By definition, the de Rham-Witt complex $W, \Omega^*_A$ is the universal Witt complex over $A$. The existence is proved in [5, theorem A], which also shows that the canonical map

$$\Omega^*_W(A) \to W, \Omega^*_A$$

is surjective. Hence, every element of $W_n, \Omega^*_A$ can be written, non-uniquely, as a sum of forms $\omega = a_0 da_1 \ldots da_q$ with $a_i \in W_n(A)$. In particular, the restriction map is surjective. For the de Rham-Witt complex, the structure map

$$\lambda: W_n(A) \sim W_n, \Omega^*_A$$

is an isomorphism, and therefore, we frequently omit it from the notation.

1.2. The definition of the ring $W_n(I)$ of Witt vectors does not require that the ring $I$ be unital.

**Lemma 1.2.1.** Let $A$ be a ring and $I \subset A$ an ideal. Then $W_n(I) \subset W_n(A)$ is an ideal and the natural projection induces an isomorphism

$$W_n(A)/W_n(I) \sim W_n(A/I).$$
Proof. Only the last statement needs proof. We argue by induction on \( n \) starting from the case \( n = 1 \) which is trivial. In the induction step we consider the 3x3-diagram with exact columns:

\[
\begin{array}{cccc}
0 & 0 & 0 \\
0 & \to & \to & \to \\
0 & \downarrow & \downarrow & \downarrow \\
W_n(I) & \to & W_n(A) & \to W_n(A/I) & \to 0 \\
R & \downarrow & \downarrow & \downarrow & R \\
W_{n-1}(I) & \to & W_{n-1}(A) & \to W_{n-1}(A/I) & \to 0 \\
0 & 0 & 0 & \downarrow & \downarrow & \downarrow \\
0 & \to & \to & \to & \to \\
0 & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \to & \to & \to & \to \\
0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

The top and bottom row are exact by induction. Hence, so is the middle row. □

The following result is [2, lemma 2.2.1]; for the convenience of the reader we include it here.

Lemma 1.2.2. Let \( I \subset A \) be an ideal, and let \( W_n \Omega^*_A(\Lambda, I) \) be the differential graded ideal of \( W_n \Omega^*_A \) generated by \( W_n(I) \subset W_n(A) \). Then the canonical projection

\[
W_n \Omega^*_A / W_n \Omega^*_A(\Lambda, I) \sim W_n \Omega^*_A(\Lambda, I)/I
\]

is an isomorphism.

Proof. We first show that \( W_n \Lambda \Omega^*_A / W_n \Omega^*_A(\Lambda, I) \) is a Witt complex over \( A/I \). The definition and lemma 1.2.1 show that it is a pro-differential graded ring with underlying pro-ring \( W_n \Lambda A \). Hence, we need only show that the operators \( F, R \) and \( V \) on \( W_n \Omega^*_A \) descend to operators on \( W_n \Omega^*_A / W_n \Omega^*_A(\Lambda, I) \), or equivalently, that

\[
\begin{align*}
RW_n \Omega^*_A(\Lambda, I) & \subset W_{n-1} \Omega^*_A(\Lambda, I), \\
FW_n \Omega^*_A(\Lambda, I) & \subset W_{n-1} \Omega^*_A(\Lambda, I), \\
VW_n \Omega^*_A(\Lambda, I) & \subset W_{n+1} \Omega^*_A(\Lambda, I).
\end{align*}
\]

The elements of \( W_n \Omega^*_A(\Lambda, I) \) are sums of forms \( \omega = a_0 da_1 \ldots da_q \), where \( a_i \in W_n(A) \), for all \( i \), and where \( a_i \in W_n(I) \), for some \( i \). The statement for the Verschiebung map then follows from the formula

\[
V(\omega) = V(a_0 FdV(a_1) \ldots FdV(a_q)) = V(a_0)dv(a_1) \ldots dv(a_q).
\]

In the case of the Frobenius map, we first note that

\[
F(\omega) = Fa_0 \cdot Fda_1 \ldots \cdot Fda_q.
\]

If \( a_0 \in W_n(I) \) then \( F(a_0) \in W_{n-1}(I) \). If \( a_i \in W_n(I) \), for some \( 1 \leq i \leq q \), we write out \( a_i \) in Witt coordinates,

\[
a_i = [a_{i,0}]_n + V[a_{i,1}]_{n-1} + \ldots + V^{n-1}[a_{i,n-1}]_1.
\]
We then have
\[ Fda_i = [a_i,0]_{n-1}^n d[a_i,0]_{n-1} + d[a_i,1]_{n-1} + dV[a_i,2]_{n-2} + \cdots + dV^{n-2}[a_i,n-1], \]
which shows that \( Fda_i \in W_{n-1} \Omega^1_{(A,I)} \). Hence, in either case \( F(\omega) \in W_{n-1} \Omega^2_{(A,I)} \).

The statement for \( R \) is clear. This shows that \( W : \Omega_A^1/W. \Omega^1_{(A,I)} \) is a Witt complex over \( A/I \). One immediately verifies that it is the universal one. \( \square \)

**Corollary 1.2.3.** The abelian group \( W_n \Omega^1_{(A,I)} \) is generated by elements of the form \( dV^s[x]_{n-s} \), \( V^s([x]_{n-s}d[a]_{n-s}) \) and \( V^s([a]_{n-s}d[x]_{n-s}) \) with \( 0 \leq s < n, a \in A, \) and \( x \in I \).

**Proof.** The group \( W_n \Omega^1_{(A,I)} \) is generated by elements of the form \( adx \) and \( xda \) with \( a \in W_n(A) \) and \( x \in W_n(I) \). Writing \( a \) and \( x \) out in Witt coordinates,
\[
\begin{align*}
a &= [a_0]_n + V[a_1]_{n-1} + \cdots + V^{n-1}[a_{n-1}]_1, \\
x &= [x_0]_n + V[x_1]_{n-1} + \cdots + V^{n-1}[x_{n-1}]_1,
\end{align*}
\]
we see that only the generators \( V^s[a]_{n-s}dV^t[x]_{n-t} \) and \( V^s[x]_{n-s}dV^t[a]_{n-t} \) with \( a \in A, x \in I, \) and \( 0 \leq s, t < n, \) are needed. The generators \( V^s[a]_{n-s} \cdot dV^t[x]_{n-t} \) with \( s \geq t \) may be rewritten
\[
V^s[a]_{n-s} \cdot dV^t[x]_{n-t} = V^s([a]_{n-s}F^s dV^t[x]_{n-t}) = V^s([a]_{n-s} [x]^{P s-1}_{P s-1} d[x]_{n-s}),
\]
which is of the desired form. And if \( s \leq t, \) we have
\[
V^s[a]_{n-s} \cdot dV^t([x]_{n-t}) = dV^s([a]_{n-s} V^t([x]_{n-t})) - dV^s([a]_{n-s}) V^t([x]_{n-t})
\]
\[= dV^t([a]^{P s-t-1}_{P s-t-1} [x]_{n-t}) - V^t([a]^{P s-t-1}_{P s-t-1} [x]_{n-t} d[a]_{n-t}). \]

Similarly, for the generators \( V^s[x]_{n-s} \cdot dV^t[a]_{n-t}. \) \( \square \)

## 2. The relative theory \( \text{TR}^n(A, I; p) \)

**2.1.** Let \( I \subset A \) be an ideal and let
\[
\text{TH}(A, I) = \text{TH}(A \to A/I)
\]
be the (cyclotomic) spectrum defined in [1, appendix]. Then for all \( n \geq 1, \) there is a natural cofibration sequence of \( \text{TR}^n(A; p) \)-module spectra
\[
\text{TR}^n(A, I; p) \to \text{TR}^n(A; p) \to \text{TR}^n(A/I; p) \to \Sigma \text{TR}^n(A, I; p).
\]
This is proved in loc. cit. under a certain connectivity requirement. But, as a consequence of [10, theorem 4.2.8], this requirement may be dropped. For the module structure we refer to [6, section 2.7]; see also [3, appendix].

**Lemma 2.1.1.** There is a canonical isomorphism \( I \xrightarrow{\sim} \text{TH}_0(A, I), \) and as an abelian group, \( \text{TH}_1(A, I) \) is generated by elements of the form \( xda \) and \( adx \) with \( a \in A \) and \( x \in I \).

**Proof.** The spectrum \( \text{TH}(A, I) \) is defined as the geometric realization of a simplicial symmetric spectrum \( [s] \mapsto \text{TH}(A, I)_s. \) The spectrum \( \text{TH}(A, I)_s \) in simplicial degree \( s \) has the homotopy type of the homotopy colimit of the punctured \( s \)-cube which to \( T \subseteq \{1, 2, \ldots, s\} \) associates the smash product
\[
A_1 \wedge A_2 \wedge \cdots \wedge A_s,
\]
where $A_i = A$ (resp. $A_i = I$) if $i \in T$ (resp. if $i \notin T$). (Here we denote a ring and its Eilenberg-MacLane spectrum by the same symbol.) It follows that in the skeleton spectral sequence
\[ E_{s,t}^1 = \pi_t(\text{TH}(A, I)_s) \Rightarrow \text{TH}_{s+t}(A, I), \]
we have
\[ E_{0,t}^1 = \begin{cases} I, & \text{if } t = 0, \\ 0, & \text{if } t > 0, \end{cases} \]
\[ E_{1,0}^1 = I \otimes A \oplus I \otimes I \otimes A \otimes I. \]
Since $A$ is commutative, the differential
\[ d^1 : E_{0,1}^1 \to E_{1,0}^1 \]
is zero, and hence, the edge-homomorphism $I \xrightarrow{\sim} \text{TH}_0(A, I)$ is an isomorphism. Finally, the elements $xda$ (resp. $adx$) with $a \in A$ and $x \in I$ are represented in the spectral sequence by $x \otimes a$ (resp. $a \otimes x$) in $E_{1,0}^1$.

Remark 2.1.2. The generators $xda$ and $adx$ of $\text{TH}_1(A, I)$ are subject to the additivity relations
\[
(a_1 + a_2)d(x_1 + x_2) = a_1dx_1 + a_1dx_2 + a_2dx_1 + a_2dx_2,
\]
\[
(x_1 + x_2)d(a_1 + a_2) = x_1da_1 + x_1da_2 + x_2da_1 + x_2da_2,
\]
where $a_1, a_2 \in A$ and $x_1, x_2 \in I$, and to the Leibniz rule
\[ a_0d(a_1a_2) = a_0a_1da_2 + a_2a_0da_1, \]
where $a_i \in A$, for all $i = 0, 1, 2$, and $a_i \in I$, for some $i = 0, 1, 2$. Indeed, the additivity relations follows from the definition of the tensor product, which gives $E_{1,0}^1$, and the Leibniz rule follows from the differential $d^1 : E_{0,1}^1 \to E_{1,0}^1$.

Since $\text{TH}(A, I)$ is cyclotomic in the sense of [6, definition 2.2], we have a natural cofibration sequence
\[ h\text{TR}^n(A, I; p) \xrightarrow{N} \text{TR}^n(A, I; p) \xrightarrow{R} \text{TR}^{n-1}(A, I; p) \xrightarrow{d} \Sigma h\text{TR}^n(A, I; p); \]
see [6, theorem 2.2]. The left hand term
\[ h\text{TR}^n(A, I; p) = \mathbb{H}_*(C_{p^{n-1}}, \text{TH}(A, I)) \]
is the group homology spectrum (or Borel construction) whose homotopy groups are the abutment of a first quadrant spectral sequence
\[ E_{s,t}^2 = H_*(C_{p^{n-1}}, \text{TH}_{s+t}(A, I)) \Rightarrow h\text{TR}^n_s(A, I; p). \]
We refer the reader to [7, paragraph 4] for a thorough treatment of the construction of this spectral sequence. The $C_{p^{n-1}}$-module $\text{TH}_1(A, I)$ is trivial, since the $C_{p^{n-1}}$-action on $\text{TH}(A, I)$ comes from a circle action.

Lemma 2.1.3. The map $I \otimes \text{TH}_1(A, I) \to h\text{TR}^n_0(A, I; p)$, which to $(x, \omega)$ associates $dV^{n-1}x + V^{n-1}\omega$, is a surjection.

Proof. The spectral sequence above amounts to an exact sequence
\[ p^{n-1}I \xrightarrow{d^2} \text{TH}_1(A, I) \xrightarrow{V^{n-1}} h\text{TR}^n_0(A, I; p) \xrightarrow{\pi} I/p^{n-1}I \to 0, \]
where $\pi$ is the edge-homomorphism to the baseline. Moreover, the composite
\[ I \xrightarrow{V^{n-1}} h\text{TR}^n_0(A, I; p) \xrightarrow{d} h\text{TR}^n_1(A, I; p) \xrightarrow{\pi} I/p^{n-1}I \]
may be identified with the map \( H_0(C^{p-1}_p, I) \to H_1(C^{p-1}_p, I) \) given by multiplication by the fundamental class \([\mathbb{T}/C^{p-1}_p]\). It is well-known that this map is an epimorphism. □

**Proposition 2.1.4.** As a non-unital ring, \( TR^n_0(A; I; p) \) is canonically isomorphic to \( W_n(I) \), and as an abelian group, \( TR^n_0(A; I; p) \) is generated by elements of the form \( dV^s([x]_{n-s}), V^s([x]_{n-s}d[a]_{n-s}) \) and \( V^s([a]_{n-s}d[x]_{n-s}) \), where \( 0 \leq s < n, a \in A, \) and \( x \in I \).

**Proof.** The first statement follows from the proof of [6, theorem F]. Indeed, it is not necessary for this proof that the ring \( I \) be unital. The second statement follows from lemmas 2.1.1 and 2.1.3 by an induction argument based on the exact sequence

\[
h TR^n_1(A; I; p) \to TR^n_1(A; I; p) \to TR^n_1(A; I; p) \to 0.
\]

The maps in the sequence commute with \( d \) and \( V \) (and \( F \)). □

**Proof of the theorem.** The statement for \( q = 0 \) is [6, theorem F], so consider \( q = 1 \). If \( A \) is a polynomial algebra over \( \mathbb{Z}(p) \), the statement was proved in [5]. In the general case, we write \( A = R/I \) with \( R \) a polynomial algebra over \( \mathbb{Z}(p) \) and consider the following diagram with exact rows

\[
\begin{array}{cccccc}
0 & \to & W_n \Omega^1_{(R, I)} & \to & W_n \Omega^1_R & \to & W_n \Omega^1_{R/I} & \to & 0 \\
& & & \downarrow \sim & \downarrow & \downarrow & \\
& & TR^n_1(R, I; p) & \to & TR^n_1(R; p) & \to & TR^n_1(R/I, p) & \to & 0
\end{array}
\]

Then the middle vertical map is an isomorphism, and hence, it suffices to show that the image of the composite

\[
W_n \Omega^1_{(R, I)} \to W_n \Omega^1_R \xrightarrow{\sim} TR^n_1(R; p)
\]

coincides with the image of the canonical map

\[
TR^n_1(R, I; p) \to TR^n_1(R; p).
\]

But corollaries 1.2.3 and 2.1.4 identifies both images with the subgroup generated by elements of the form \( dV^s([x]_{n-s}), V^s([x]_{n-s}d[a]_{n-s}) \) and \( V^s([a]_{n-s}d[x]_{n-s}) \), where \( 0 \leq s < n, a \in R, \) and \( x \in I \). This concludes the proof. □

**References**


*Massachusetts Institute of Technology, Cambridge, Massachusetts*

*E-mail address: larsh@math.mit.edu*