Here we want to see to what extent the results from Doeblin’s theory for finite state spaces extend to the case when the state space is countably infinite. Throughout we will use $\pi$ to denote the row vector whose $j$th component is $\pi_j = (\mathbb{E}[\rho_j | X_0 = j])^{-1}$.

**Theorem (A).** If $j$ is transient, then $\pi_j = 0 = \lim_{n \to \infty} (P^n)_{ij}$ for all $i$.

**Proof.** Because $\mathbb{P}(\rho_j = \infty | X_0 = j) > 0$, it is clear that $\pi_j = 0$. Thus, all that we have to do is check that $\lim_{n \to \infty} (P^n)_{ij} = 0$ for all $i$. But this would follow if we show that $\sum_{n=0}^{\infty} (P^n)_{ij} < \infty$ for all $i$. But, referring to (3) in the note about the Classification of States, one sees that

$$\sum_{n=0}^{\infty} (P^n)_{ij} = \mathbb{E}[T_j | X_0 = i] \leq \frac{1}{\mathbb{P}(\rho_j = \infty | X_0 = j)} < \infty.$$ 

□

Theorem (A) takes care of the transient case. Just as in the Doeblin theory, where we had to resort to averaging in order to get a general result, here too we will have to use “convergence enhancing” techniques. Specifically, we will want to a little know about *Abel convergence*.

**Lemma 1 (Abel).** If $x_n \longrightarrow x$ in $\mathbb{R}$, then

$$\lim_{s \to 1} (1 - s) \sum_{n=0}^{\infty} s^n x_n = x.$$ 

**Proof.** By replacing $x_n$ with $x_n - x$, reduce to the case when $x_n \longrightarrow 0$. But then, for any $N \geq 1$ and $s \in (0, 1)$,

$$\left| (1 - s) \sum_{n=0}^{\infty} s^n x_n \right| \leq \max_{1 \leq n \leq N} |x_n| (1 - s) \sum_{n=0}^{N-1} s^n + \sup_{n \geq N} |x_n| \sum_{n=N}^{\infty} s^n.$$ 

Thus, for each $N \geq 1$,

$$\lim_{s \to 1} \left| (1 - s) \sum_{n=0}^{\infty} s^n x_n \right| \leq \sup_{n \geq N} |x_n|. \quad \square$$ 

In view of Lemma 1, it is reasonable to say that $\{x_n\}_0^{\infty}$ is *Abel convergent* to $x$ if

$$(1 - s) \sum_{n=0}^{\infty} s^n x_n \longrightarrow x \quad \text{as} \quad s \nearrow 1.$$ 

With this in mind, we now set

$$P(s) = \sum_{n=0}^{\infty} s^n P^n \quad \text{and} \quad R(s) = (1 - s)P(s) \quad \text{for} \quad s \in [0, 1).$$

Notice that, for each $s \in [0, 1)$, $R(s)$ is a transition probability matrix.

Our next goal is to prove the following statement about the Abel convergence $\{(P^n)_{ij} : n \geq 0\}$ to $\pi_j$.

**Theorem (B).** Given a any pair $(i, j)$, $\lim_{s \to 1} R(s)_{ij} = \mathbb{P}(\rho_j | X_0 = i) \pi_j$.

In proving Theorem (B), as well the results below, it will be useful to make reference to the easily derived facts summarized in the following lemma.
Lemma 1. For each \( m \in \mathbb{N} \), suppose \( a_m : [0,1] \rightarrow [0,\infty) \) is given, and assume that \( a_m \equiv \lim_{s \downarrow 1} a_m(s) \) exists. Then

\[
\sum_{m=0}^{\infty} a_m \leq \lim_{s \downarrow 1} \sum_{m=0}^{\infty} a_m(s).
\]

Furthermore, if, for each \( m \), \( a_m(s) \) is a non-decreasing function of \( s \) or if

\[
\sum_{m=0}^{\infty} \sup_{s \in [0,1]} a_m(s) < \infty,
\]

then

\[
\sum_{m=0}^{\infty} a_m = \lim_{s \downarrow 1} \sum_{m=0}^{\infty} a_m(s).
\]

Proof of Theorem (B). Notice that, for any \((i, j)\) and \( n \geq 1 \),

\[
(P^n)_{ij} = \sum_{m=1}^{n} P(X_n = j \& \rho_j = m \mid X_0 = i) = \sum_{m=1}^{n} (P^{n-m})_{jj}P(\rho_j = m \mid X_0 = i).
\]

Thus, for all \((i, j)\),

\[
(1) \quad (P^n)_{ij} \sum_{m=1}^{n} f(m)_{ij} (P^{n-m})_{jj}, \quad n \geq 1, \text{ where } f(n)_{ij} \equiv P(\rho_j = n \mid X_0 = i).
\]

In particular, if

\[
(2) \quad F(s)_{ij} \equiv \sum_{n=1}^{\infty} s^n f(n)_{ij} \text{ for } s \in [0, 1),
\]

then it is an easy matter to pass from (1) to the renewal equation

\[
(3) \quad P(s)_{ij} = \delta_{ij} + F(s)_{ij} P(s)_{ij}, \quad \text{for all } s \in [0, 1) \text{ and } (i, j).
\]

Given any \( j \), (3) combined with Lemma 1 say that

\[
R(s)_{jj} = \frac{1-s}{1-F(s)_{jj}} = \left( \sum_{n=1}^{\infty} \frac{1-s^n}{1-s} f(n)_{jj} \right)^{-1} \backslash \left( \sum_{n=1}^{\infty} n f(n)_{jj} \right)^{-1} = \pi_j,
\]

since \( \frac{1-s^n}{1-s} \not\sim n \) as \( s \not\sim 1 \). Moreover, if \( i \neq j \), then the preceding and another application of (3) show that

\[
R(s)_{ij} = F(s)_{ij} R(s)_{jj} \rightarrow \sum_{m=1}^{\infty} P(\rho_j = m \mid X_0 = i) \pi_j = P(\rho_j < \infty \mid X_0 = i) \pi_j. \quad \Box
\]

Before applying the preceding to the study of \( P \)-stationary probability vectors, we make the following easy remark.
Lemma 2. If $Q$ is a transition probability matrix and $\nu$ is a vector with $||\nu||_1 < \infty$, then $\nu Q = \nu$ if and only if either $\nu Q \leq \nu$ or $\nu \geq Q \nu$.

Proof. Suppose $\nu \leq \nu Q$ but that $\nu \neq \nu Q$. Then
\[ \sum_j \nu_j < (\nu Q)_j = \sum_i \nu_i Q_{ij} = \sum_i \nu_i. \]
Similarly, $\nu Q \leq \nu$ but $\nu Q \neq \nu$ leads to $\sum_i \nu < \sum_j \nu_j$. $\square$

Theorem (C). Suppose that $\mu$ is a $P$-stationary probability vector. Then, for all $(i,j)$,
\[ \mu_j = \left( \sum_{\{i: i \to j\}} \mu_i \right) \pi_j \text{ for all } i. \]
Further, if $j$ is recurrent, $C = \{i : i \to j\}$, and
\[ \pi_i^C = \begin{cases} \pi_i & \text{for } i \in C \\ 0 & \text{for } i \notin C \end{cases}, \]
then $\pi_j = 0 \implies \pi_i^C = 0$, $\pi_j > 0 \implies \pi_i^C$ is a probability vector, and, in either case, $\pi_i^C = \pi_i^C P$. In particular, for any $j$, $\pi_j = 0 \implies \mu_j = 0$ for every $P$-stationary vector $\mu$ and $\pi_j > 0$ implies that $\pi_i > 0$ whenever $i \in C \equiv \{i : i \to j\}$ and that (cf. (5)) $\pi_i^C$ is the unique $P$-stationary vector which vanishes off $C$.

Proof. First suppose that $\mu$ is a $P$-stationary probability vector. Then $\mu = \mu R(s)$, and so, by Theorem (B) and Lemma 1, we find that
\[ \mu_j = \sum_i \mu_i R(s)_{ij} \to \sum_i \mu_i P(\rho_j < \infty | X_0 = i) \pi_j. \]
If $j$ is transient, then $\pi_j = 0$ and so there is nothing more to do. If $j$ is recurrent, then, by Lemma 2 in the Classification notes, $\mu_i P(\rho_j < \infty | X_0 = i) = 0$ if $i \not\to j$ and $P(\rho_j < \infty | X_0 = i) = 1$ if $i \to j$. Hence, (4) is proved.

Turning to the second assertion, assume that $j$ is recurrent, and set $C = \{i : i \to j\}$. Then $j$ is, by Lemma 2 in the Classification notes, for each $i \in C$ and $s \in [0,1)$, $R(s)_{ik} > 0$ if and only if $k \in C$. Next note that, by Theorem (B) and Lemma 1 above, for any $i$,
\[ \sum_i \pi_i^C \leq \lim_{s \nearrow 1} \sum_i R(s)_{ji} = 1 \]
and
\[ (\pi_i^C P)_i = \sum_{k \in C} \pi_k P_{ki} \leq \lim_{s \nearrow 1} \sum_{k \in C} R(s)_{jk} P_{ki} = \pi_i^C, \]
since
\[ \sum_k R(s)_{jk} P_{ki} = \frac{R(s) - (1-s)\delta_{jk}}{s} \to P(\rho_i | X_0 = j) \pi_i = \pi_i^C \text{ as } s \nearrow 1. \]
Hence, by Lemma 2, $\pi_i^C = \pi_i^C P$. In particular, if $\pi_j = 0$, then, because $\pi_i^C = \pi_n^C P^n$ for all $n \in \mathbb{N}$ and, for each $i \in C$, $(P^n)_{ij} > 0$ for some $n$, we would have that $\pi_i = 0$ for all $i \in C$. On the other hand, if $\pi_j > 0$, then by Theorem (B) and Lemma 1,
\[ 0 < \pi_j = \sum_{i \in C} \pi_i R(s)_{ij} \to \sum_{i \in C} \pi_j = \left( \sum_{i \in C} \pi_i \right) \pi_j, \]
and so $\sum_{i \in C} \pi_i = 1$.

Finally, the concluding uniqueness assertion is now an immediate consequence of (4). $\square$

---

$^1$Given two vectors $\alpha$ and $\beta$, we write $\alpha \leq \beta$ if $\alpha_i \leq \beta_i$ for all $i$. 
Proof. To prove the first assertion, notice that
\[ \lim_{n \to \infty} H_n = \pi \]
Hence, by Lemma 3,
\[ \pi \]
Similarly,
\[ \pi \]
Finally, by Lemma 3, we have proved (6) in general.
\[ \square \]

**Corollary.** There is \( \mathbf{P} \)-stationary \( \mu \) with \( \mu_j > 0 \) if and only if \( \pi_j > 0 \). Moreover, if \( \mu \) exists, then
\[ \mu_i = \left( \sum_{k=1}^{m} \mu_k \right) \pi_i > 0 \text{ when } i \to j. \]

**A Small Improvement**
In this section we will see how to pass from Theorem (B) to the conclusion that

(6) \[ \lim_{N \to \infty} (A_N)_{ij} = \mathbb{P}(\rho_j | X_0 = i) \pi_j \text{ where } A_N \equiv \frac{1}{N} \sum_{n=0}^{N-1} \mathbf{P}^n. \]

**Lemma 3.** Given any \( \{a_m : m \geq 0\} \subset [0,1] \) and \( A_n = \frac{1}{n} \sum_{\ell=0}^{n-1} a_\ell \), then \( |A_n - A_{n-m}| \leq \frac{m}{n} \) for all \( n \geq m \geq 1 \).

**Proof.** Clearly,
\[ A_n - A_{n-m} = \frac{1}{n} \sum_{\ell=n-m}^{n-1} a_\ell - \frac{m}{n(n-m)} \sum_{\ell=0}^{n-m-1} a_\ell, \]
which leads to \( -\frac{m}{n} \leq A_n - A_{n-m} \leq \frac{m}{n} \). \( \square \)

**Lemma 4.** For all \((i,j), \lim_{N \to \infty} (A)_{ij} \leq c \pi_j \). In addition, for any \( j \) and any subsequence \( \{N_\ell : \ell \geq 0\} \subseteq \mathbb{N}, \lim_{\ell \to \infty} (A)_{jj} = c \pi_j \implies \lim_{\ell \to \infty} (A)_{ij} = \mathbb{P}(\rho_j < \infty | X_0 = i) \alpha_{ij} \text{ for all } i \neq j. \)

**Proof.** To prove the first assertion, notice that
\[ (A_N)_{ij} \leq \frac{1}{N} (1 - \frac{1}{N})^{-N} \sum_{n=0}^{N-1} (1 - \frac{1}{N})^n (\mathbf{P}^n)_{ij} \leq (1 - \frac{1}{N})^{-N} \mathbf{R}(1 - \frac{1}{N}). \]
Hence, since \( \lim_{s \to 1} \mathbf{R}(s)_{ij} = \mathbb{P}(\rho_j < \infty | X_0 = i) \pi_j \leq \pi_j \), we are done.

To handle the second part, start from (3) and conclude that
\[ (A_N)_{ij} = \sum_{m=1}^{N} \mathbb{P}(\rho_j = m | X_0 = i) (1 - \frac{m}{N}) (A_{N-m})_{jj}. \]
Hence, by Lemma 3,
\[ \left| (A_N)_{ij} - \mathbb{P}(\rho_j \leq N | X_0 = i) \alpha \right| \leq \sum_{m=1}^{N} \mathbb{P}(\rho_j = m | X_0 = i) \left( \frac{m}{N} + \left| (A_{N-m})_{jj} - \alpha \right| \right) \]
\[ \leq \frac{2}{N} \sum_{m=1}^{N} m \mathbb{P}(\rho_j = m | X_0 = i) + \left| (A_N)_{jj} - \alpha \right|. \]
Applying this to \( N_\ell \) and letting \( \ell \to \infty \), we get the desired result. \( \square \)

**Proof of (6).** If \( \pi_j = 0 \), then \( \lim_{N \to \infty} (A_N)_{ij} = 0 = \mathbb{P}(\rho_j < \infty | X_0 = i) \pi_j \) for all \( i \). If \( \pi_j > 0 \), then \( j \) is recurrent and, if \( C = \{i : i \to j\} \), (cf. (5)) \( \pi^C \) is a \( \mathbf{P} \)-stationary probability vector. Hence, \( \pi_j \sum_{i \in C} \pi_i (A)_{ij} \) for all \( N \geq 1 \). In particular, if \( \{N_\ell : \ell \geq 0\} \) is a subsequence for which \( \lim_{\ell \to \infty} (A)_{jj} = \alpha^+ \equiv \lim_{N \to \infty} (A)_{jj} \), then, by Lemma 3, \( \lim_{\ell \to \infty} (A)_{ij} = \alpha^+ \) for all \( i \in C \) and so, by Lemma 1,
\[ \pi_j = \lim_{\ell \to \infty} \sum_{i \in C} \pi_i (A_{N_\ell})_{ij} = \alpha^+ \sum_{i \in C} \pi_i = \alpha^+. \]
Similarly, \( \pi_j = \alpha^- \equiv \lim_{N \to \infty} (A)_{jj} \), and therefore we now know that \( \pi_j = \lim_{N \to \infty} (A_N)_{jj} \). Finally, by Lemma 3, we have proved (6) in general. \( \square \)
Corollary (Mean Ergodic). If \( \pi_j > 0 \), then

\[
P(X_0 \rightarrow j) = 1 \implies \lim_{n \to \infty} \mathbb{E} \left[ \left( \frac{1}{n} \sum_{m=0}^{n-1} 1_{\{j\}}(X_m) - \pi_j \right)^2 \right] = 0.
\]

Proof. Set \( C = \{ i : i \rightarrow j \} \). Then \( P(X_m \in C \text{ for all } m \geq 0) = 1 \). Hence, without loss in generality, we will assume that \( i \rightarrow j \) for all \( i \), in which case \( \pi_i > 0 \) for all \( i \). In particular, since

\[
\mathbb{E} \left[ \left( \frac{1}{n} \sum_{m=0}^{n-1} 1_{\{j\}}(X_m) - \pi_j \right)^2 \mid X_0 = i \right] \leq \frac{1}{\pi_i} \sum_k \pi_k \mathbb{E} \left[ \left( \frac{1}{n} \sum_{m=0}^{n-1} 1_{\{j\}}(X_m) - \pi_j \right)^2 \mid X_0 = k \right],
\]

it suffices to prove that

\[
\lim_{n \to \infty} \mathbb{E} \left[ \left( \frac{1}{n} \sum_{m=0}^{n-1} 1_{\{j\}}(X_m) - \pi_j \right)^2 \right] = 0 \quad \text{when } P(X_0 = i) = \pi_i \text{ for all } i.
\]

But, just as in the proof of Theorem (C) from the Doeblin theory notes,

\[
\mathbb{E} \left[ \left( \frac{1}{n} \sum_{m=0}^{n-1} 1_{\{j\}}(X_m) - \pi_j \right)^2 \right] \leq \frac{2}{n^2} \sum_{k=0}^{n-1} (n - k) \mathbb{E} [f_{X_k}(A_{n-k}f)_{X_k}],
\]

where \( f_i = 1_{\{j\}}(i) - \pi_j \). Since \( \pi \) is \( P \)-stationary, when \( \pi \) is the distribution of \( X_0 \), \( \mathbb{E} [f_{X_k}(A_{n-k}f)_{X_k}] = \pi(f_{A_{n-k}f}) \), and so the preceding becomes

\[
\mathbb{E} \left[ \left( \frac{1}{n} \sum_{m=0}^{n-1} 1_{\{j\}}(X_m) - \pi_j \right)^2 \right] \leq \frac{2}{n^2} \sum_{m=0}^{n-1} m \pi(f_{A_m f}).
\]

Finally, because, by (5) and Lemma 1, for each \( \epsilon > 0 \) there exists an \( N_\epsilon \in \mathbb{Z}^- \) such that

\[
|\pi(f_{A_m f})| \leq \sum_i \pi_i |f_i|(A_m f)_i < \epsilon \quad \text{for } m \geq N_\epsilon,
\]

we find that

\[
\lim_{n \to \infty} \mathbb{E} \left[ \left( \frac{1}{n} \sum_{m=0}^{n-1} 1_{\{j\}}(X_m) - \pi_j \right)^2 \right] \leq \lim_{n \to \infty} \frac{2}{n^2} \sum_{m=0}^{N_\epsilon-1} m |\pi(f_{A_m f})| + \lim_{n \to \infty} \frac{2\epsilon}{n^2} \sum_{m=N_\epsilon}^{n} m \leq \epsilon. \quad \square
\]

Refinement in the Aperiodic Case

In general, we know that the conclusion drawn in (6) is optimal. Nonetheless, the following statement provides a significant sharpening in those situations to which it applies.

Theorem (D). If \( j \) is either transient or aperiodic, then

\[
\lim_{n \to \infty} (P^n)_{ij} = P(\rho - j < \infty | X_0 = i) \pi_j \quad \text{for all } i.
\]

The case when \( j \) is transient is covered by Theorem (A). Thus, we need only worry about the case when \( j \) is recurrent and aperiodic.
Lemma 5. If $j$ is aperiodic, then there exists an $N(j) \geq 1$ such that (cf. the notation introduced in the Classification notes) $\max_{1 \leq m \leq n} \mathbb{P}(\rho_j^{(m)} = n|X_0 = j) > 0$ for all $n \geq N(j)$.

Proof. By the Corollary to Theorem (A) in the Classification notes, we know that there is an $N(j)$ such that $(P^n)_{jj} > 0$ for all $n \geq N(j)$. Hence, since, for any $n \geq 1$,

$$(P^n)_{jj} = \sum_{m=1}^{n} \mathbb{P}(\rho_j^{(m)} = n \land X_n = j|X_0 = j) \leq \sum_{m=1}^{n} \mathbb{P}(\rho_j^{(m)} = n|X_0 = j),$$

we are done. □

Lemma 6. Assume that $j$ is aperiodic and recurrent, and set $\alpha_j^- = \lim_{n \to \infty}(P^n)_{jj}$ and $\alpha_j^+ = \lim_{n \to \infty}(P^n)_{jj}$. Then there exist subsequences $\{n_\ell^- : \ell \geq 1\}$ and $\{n_\ell^+ : \ell \geq 1\}$ such that

$$\alpha_j^+ = \lim_{\ell \to \infty}(P^{n_\ell^+-r})_{jj} \text{ for all } r \geq 0.$$

Proof. Choose the subsequence $\{n_\ell^- : \ell \geq 1\}$ so that $(P^{n_\ell^-})_{jj} \to \alpha_j^+$, and, using Lemma 5, choose $N \geq 1$ so that $\max_{1 \leq m \leq n} \mathbb{P}(\rho_j^{(m)} = n|X_0 = j) > 0$ for all $n \geq N$. Given $r \geq N$, $M \geq 1$, and $\ell$ such that $n_\ell \geq M + r$, choose $1 \leq m \leq r$ so that $\delta = \mathbb{P}(\rho_j^{(m)} = r|X_0 = j) > 0$, and observe that

$$(P^{n_\ell^-})_{jj} = \mathbb{P}(X_{n_\ell} = j \land \rho_j^{(m)} = r \mid X_0 = j) + \mathbb{P}(X_{n_\ell} = j \land \rho_j^{(m)} \neq r \mid X_0 = j)
= \delta(P^{n_\ell^-})_{jj} + \mathbb{P}(X_{n_\ell} = j \land \rho_j^{(m)} \neq r \mid X_0 = j) + \mathbb{P}(X_{n_\ell} = j \land \rho_j^{(m)} > n_\ell - M \mid X_0 = j).$$

Furthermore,

$$\mathbb{P}(X_{n_\ell} = j \land \rho_j^{(m)} \neq r \mid X_0 = j) = \sum_{k=1}^{n_\ell-M} \mathbb{P}(\rho_j^{(m)} = k \mid X_0 = j)(P^{n_\ell-k})_{jj} \leq (1 - \delta) \sup_{n \geq M} (P^n)_{jj},$$

while

$$\mathbb{P}(X_{n_\ell} = j \land \rho_j^{(m)} > n_\ell - M \mid X_0 = j) \leq \mathbb{P}(\rho^{(m)} > n_\ell - M \mid X_0 = j).$$

Hence, since $j$ is recurrent and therefore $\mathbb{P}(\rho_j^{(m)} < \infty \mid X_0 = j) = 1$, we get

$$\alpha_j^+ \leq \delta \lim_{\ell \to \infty} (P^{n_\ell^-})_{jj} + (1 - \delta) \sup_{n \geq M} (P^n)_{jj},$$

after letting $\ell \to \infty$. Since this is true for all $M \geq 1$, it leads to

$$\alpha_j^+ \leq \delta \lim_{\ell \to \infty} (P^{n_\ell^-})_{jj} + (1 - \delta)\alpha_j^+,\,$$

from which we see that $\lim_{\ell \to \infty} (P^{n_\ell^-})_{jj} \geq \alpha_j^+$. But obviously $\lim_{\ell \to \infty} (P^{n_\ell^-})_{jj} \leq \alpha_j^+$, and so we have now shown that $\lim_{\ell \to \infty} (P^{n_\ell^-})_{jj} = \alpha_j^+$ for all $r \geq N$. Now choose $L$ so that $n_L \geq N$, take $n_\ell^- = n_{\ell + L} - N$, and conclude that $\lim_{\ell \to \infty} (P^{n_\ell^-})_{jj} = \alpha_j^+$ for all $r \geq 0$.

The construction of $\{n_\ell^- : \ell \geq 1\}$ is essentially the same and is left as an exercise. □
Lemma 7. If \( j \) is aperiodic and recurrent, then \( \lim_{n \to \infty} (P^n)_{jj} \leq \pi_j \). Furthermore, if the subsequences \( \{n^\pm_k : \ell \geq 1\} \) are the ones described in Lemma 6, then \( \lim_{\ell \to \infty} (P^{n^\pm_k})_{ij} = \alpha^\pm_{ij} \) for any \( i \) with \( i \neq j \).

Proof. Note that, for any \( n \geq 1 \),
\[
(P^n)_{jj} = \sum_{r=1}^{n} \mathbb{P}(\rho_j = r \mid X_0 = j)(P^{n-r})_{jj}
\]
\[
= \sum_{r=1}^{n} \mathbb{P}(\rho_j \geq r \mid X_0 = j)(P^{n-r})_{jj} - \sum_{r=1}^{n} \mathbb{P}(\rho_j \geq r + 1 \mid X_0 = j)(P^{n-r})_{jj}
\]
\[
= \sum_{r=1}^{n} \mathbb{P}(\rho_j \geq r \mid X_0 = j)(P^{n-r})_{jj} - \sum_{r=2}^{n+1} \mathbb{P}(\rho_j \geq r \mid X_0 = j)(P^{n-r})_{jj},
\]
and so, since \( \mathbb{P}(\rho_j \geq 1 \mid X_0 = j) = 1 \),
\[
\sum_{r=1}^{n+1} \mathbb{P}(\rho_j \geq r \mid X_0 = j)(P^{n+1-r})_{jj} = \sum_{r=1}^{n} \mathbb{P}(\rho_j \geq r \mid X_0 = j)(P^{n-r})_{jj}
\]
for all \( n \geq 1 \). But \( \sum_{r=1}^{n} \mathbb{P}(\rho_j \geq r \mid X_0 = j)(P^{n-r})_{jj} = 1 \) when \( n = 1 \), and so we have now proved that
\[
\sum_{r=1}^{N} \mathbb{P}(\rho_j \geq r \mid X_0 = j)(P^{n-r})_{jj} \leq \sum_{r=1}^{n} \mathbb{P}(\rho_j \geq r \mid X_0 = j)(P^{n-r})_{jj} = 1
\]
for all \( n \geq N \geq 1 \). Now use the notation and results in Lemma 6 to conclude that
\[
\alpha^+_j \sum_{r=1}^{N} \mathbb{P}(\rho_j \geq r \mid X_0 = j) = \lim_{\ell \to \infty} \sum_{r=1}^{N} \mathbb{P}(\rho_j \geq r \mid X_0 = j)(P^{n^\ell-r})_{jj} \leq 1
\]
for all \( N \geq 1 \). But this means that
\[
\alpha^+_j \mathbb{E}[\rho_j \mid X_0 = j] = \alpha^+_j \sum_{r=1}^{\infty} \mathbb{P}(\rho_j \geq r \mid X_0 = j) \leq 1,
\]
which is equivalent to the first assertion.

To prove the second assertion, simply note that
\[
(P^{n^\pm_k})_{ij} = \sum_{r=1}^{n^\pm_k} \mathbb{P}(\rho_j = r \mid X_0 = i)(P^{n^\pm_k-r})_{jj} \longrightarrow \mathbb{P}(\rho_j < \infty \mid X_0 = i)\alpha^\pm_{ij}. \quad \square
\]

Proof of Theorem (D). Assume that \( j \) is recurrent and aperiodic. By Lemma 7, we know that \( \lim_{n \to \infty} (P^n)_{jj} = 0 \) if \( \pi_j = 0 \). Thus, if \( \pi_j = 0 \), then, for any \( i \), Lemma 1 justifies
\[
\lim_{n \to \infty} (P^n)_{ij} = \lim_{n \to \infty} \sum_{r=1}^{n} \mathbb{P}(\rho_j = r \mid X_0 = j)(P^{n-r})_{jj} = 0 = \mathbb{P}(\rho_j < \infty \mid X_0 = i)\pi_j.
\]

In order to handle the case when \( \pi_j > 0 \), take \( C = \{i : i \neq j\} \), and take \( \pi^C \) as in (5). Then, by the last part of Theorem (C), \( \pi^C \) is a \( P \)-stationary probability vector. In particular, by the last part of Lemma 7, Lemma 1 justifies
\[
\pi_j = \sum_{i \in C} \pi_i (P^{n^\pm_k})_{ij} \longrightarrow \alpha^\pm_j \sum_{i \in C} \pi_i \alpha^\pm_j = \alpha^\pm_j,
\]
and so \( (P^{n^\pm_k})_{jj} \longrightarrow \pi_j \). Finally, if \( i \neq j \), then another application of Lemma 1 yields
\[
(P^n)_{ij} = \sum_{r=1}^{n} \mathbb{P}(\rho_j = r \mid X_0 = i)(P^{n-r})_{jj} \longrightarrow \mathbb{P}(\rho_j < \infty \mid X_0 = i)\pi_j. \quad \square
\]