18.440 Exam #2, November 5th, 2004

(1) Suppose that $X$ and $Y$ are binomial random variables with, respectively, parameters $(m, p)$ and $(n, p)$. That is

$$
P(X = k) = \binom{m}{k} p^k (1 - p)^{m-k} \text{ for } 0 \leq k \leq m
$$

$$
P(Y = \ell) = \binom{n}{\ell} p^\ell (1 - p)^{n-\ell} \text{ for } 0 \leq \ell \leq n.
$$

Assuming that $Y$ is independent of $X$, compute the probability mass function for $Z = X + Y$. That is, what is the $P(Z = z)$?

**Hint:** By finding a model in which $X$, $Y$, and $Z$ arise, see if you can do this problem without any computation.

(2) Suppose a random variable $X$ admits the density function

$$
f(x) = \begin{cases} 
\frac{1}{\pi (1 - x^2)^{\frac{1}{2}}} & \text{for } x \in (-1, 1) \\
0 & \text{for } x \notin (-1, 1).
\end{cases}
$$

Compute the variance $\text{Var}(X)$ of $X$.

**Hint:** Try trigonometric substitution.

(3) Suppose that a fair coin is tossed once every minute until a head appears, at which instant a one minute egg timer is set and allowed to run until it rings. Assuming that the amount of time for which the timer is set is uniformly distributed and independent of the coin, what is the distribution of the length of time that elapses between the first toss of the coin and the instant at which the egg timer rings?
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(1) $X$ and $Y$ can be thought of, respectively, as the number of heads which occur when a $p$-coin is tossed $m$ and $n$ times. Thus, $X + Y$ can be thought of as the number of heads which occur when a $p$-coin is tossed $m + n$ times. Hence, $Z$ is a binomial with parameters $(m + n, p)$. That is,

$$P(Z = j) = \binom{m + n}{j} p^j (1 - p)^{m+n-j} \quad \text{for } 0 \leq j \leq m + n.$$ 

Alternatively, $Z$ takes its values in $\{0, \ldots, m + n\}$ and, for $0 \leq j \leq m + n$,

$$P(Z = j) = \sum_{k=0}^{m} P(Y = j - k) P(X = k)$$

$$= \sum_{k=0}^{\min\{m,j\}} \binom{n}{j-k} \binom{m}{k} p^{j-k} (1 - p)^{n-j+k} p^k (1 - p)^{m-k}$$

$$= p^j (1 - p)^{m+n-j} \sum_{k=0}^{\min\{m,j\}} \binom{n}{j-k} \binom{m}{k},$$

and

$$\binom{m + n}{j} = \sum_{k=0}^{\min\{m,j\}} \binom{n}{j-k} \binom{m}{k}$$

since one can choose a $j$ element subset from an $m + n$ element set by first segregating the $m + n$ into subsets $A$ and $B$ of sizes $m$ and $n$ and then, for each $0 \leq k \leq \min\{m, j\}$, choosing a $k$ element subset from $A$ and a $j - k$ element subset from $B$.

(2) First note that

$$E[X] = \frac{1}{\pi} \int_{-1}^{1} \frac{x}{\sqrt{1 - x^2}} \, dx = 0$$

by symmetry. Thus,

$$\text{Var}(X) = E[X^2] = \frac{1}{\pi} \int_{-1}^{1} \frac{x^2}{\sqrt{1 - x^2}} \, dx = \frac{2}{\pi} \int_{0}^{1} \frac{x^2}{\sqrt{1 - x^2}} \, dx,$$

again by symmetry. Finally, take $x = \sin t$, and conclude that

$$\text{Var}(X) = \frac{2}{\pi} \int_{0}^{\pi} \sin^2 t \, dt = \frac{2}{\pi} \int_{0}^{\pi} \frac{1 - \cos 2t}{2} \, dt = \frac{1}{2}.$$ 

(3) Let $F$ be the number of the toss on which the first head occurs. Then $P(F = n) = 2^{-n}$ for $n \geq 1$. Next, let $U$ be the length of time for which the egg timer is set. Then $U$ is independent of $N$ and $U$ is uniformly distributed on $[0, 1]$. Finally, let $T$ be the time that elapses between the first toss and the ring of the egg timer. Then $T = N - 1 + U$. Hence, $T \geq 0$ and, if $n \geq 1$ and $n - 1 \leq t \leq n$, then

$$P(n - 1 < T \leq t) = P(N = n) P(U \leq t - n + 1) = 2^{-n}(t - n + 1).$$

Hence, if $n \geq 1$ and $n - 1 \leq t \leq n$, then

$$P(T \leq t) = \sum_{m=1}^{n-1} 2^{-m} + 2^{-n} (t - n + 1) = 1 - 2^{-n+1} + 2^{-n}(t - n + 1) = 1 + 2^{-n}(t - n - 1),$$

where it is understood that $\sum_{m=1}^{0} 2^{-m} = 0$. 