Problem #2, p. 290:

(a) \( \mathbb{P}(X_1 = 0, X_2 = 0) = \frac{5 \times 4}{13 \times 12} \), \( \mathbb{P}(X_1 = 0, X_2 = 1) = \frac{5 \times 7}{13 \times 12} = \mathbb{P}(X_1 = 1, X_2 = 0) \), \( \mathbb{P}(X_1 = 1, X_2 = 1) = \frac{5 \times 4}{13 \times 12} \), and \( \mathbb{P}(X_1 = x_1, X_2 = x_2) = 0 \) for all other \((x_1, x_2)\).

(b) \( \mathbb{P}(X_3 = 0|X_1 = 0 & X_2 = 0) = \frac{6}{11} \). \( \mathbb{P}(X_3 = 0|X_1 = 0 & X_2 = 1) = \frac{4}{11} = \mathbb{P}(X_3 = 0|X_1 = 1 & X_2 = 0) \), \( \mathbb{P}(X_3 = 1|X_1 = 0 & X_2 = 0) = \frac{5}{11} \), \( \mathbb{P}(X_3 = 1|X_1 = 1 & X_2 = 0) = \frac{5}{11} \), and \( \mathbb{P}(X_3 = x_3|X_1 = x_1 & X_2 = x_2) = 0 \) for all other \((x_1, x_2, x_3)\). By combining these with the results in (a), one can now compute the \( \mathbb{P}(X_1 = x_1, X_2 = x_2, X_3 = x_3) \) via Bayes formula.

Problem #7, p. 290: Let \( T_i \) be the time of the \( i \)th success. Then \( X_1 = T_1 - 1 \) and \( X_2 = T_2 - 2 \). Moreover, \( \mathbb{P}(T_1 = m) = pq^{m-1} \) for \( m \geq 1 \) and \( \mathbb{P}(T_2 = n|T_1 = m) = pq^{n-1} - m \) for \( 1 \leq m < n \). Hence,

\[
\mathbb{P}(X_1 = n_1 & X_2 = n_2) = \mathbb{P}(T_1 = n_1 + 1 & T_2 = n_2 + 2) = pq^{n_2 + 2 - n_1 - 1}pq^{n_1 - 1} = p^2q^{n_2 - n_1}
\]

for all integers \( 0 \leq n_1 < n_2 \), and \( \mathbb{P}(X_1 = x_1 & X_2 = x_2) = 0 \) for all other \((x_1, x_2)\).

Problem #9, p. 290 \( f_{X,Y}(x,y) = \frac{6}{7}(x^2 + \frac{xy}{2}) \) for \((x,y) \in (0,1) \times (0,2) \) and 0 elsewhere.

(b) \( f_X(x) = 0 \) for \( x \notin (0,1) \) and

\[
f_X(x) = \int f_{X,Y}(x,y) \, dy = \frac{6}{7} \int_0^2 \left( x^2 + \frac{xy}{2} \right) \, dy = \frac{6}{7} \left( 2x^2 + x \right).
\]

(a) \( \int \int f_{X,Y}(x,y) \, dx \, dy = \int f_X(x) \, dx = \frac{4}{7} + \frac{3}{7} = 1. \)

(c) \( \mathbb{P}(X > Y) = \int \int f_{X,Y}(x,y) \, dx \, dy = \frac{6}{7} \int_0^1 \left( \int_0^x \left( x^2 + \frac{xy}{2} \right) \, dx \right) \, dy = \frac{6}{7} \times 54 \int_0^1 x^3 \, dx = \frac{15}{56}. \)

(d) Because

\[
\mathbb{P}(X < \frac{1}{2}) = \int_{x < \frac{1}{2}} f_X(x) \, dx = \frac{6}{7} \int_{0}^{\frac{1}{2}} \left( 2x^2 + x \right) \, dx = \frac{2}{7}
\]

and

\[
\mathbb{P}(Y > \frac{1}{2} \& X < \frac{1}{2}) = \int \int_{x < \frac{1}{2}} f_{X,Y}(x,y) \, dx \, dy = \frac{6}{7} \int_0^{\frac{1}{2}} \left( \int_{x}^{\frac{1}{2}} \left( x^2 + \frac{xy}{2} \right) \, dx \right) \, dy \\
= \frac{6}{7} \int_0^{\frac{1}{2}} \left( \frac{3}{2}x^2 + \frac{15}{16}x \right) \, dx = \frac{69}{7 \times 64}
\]

\( \mathbb{P}(Y > \frac{1}{2} \mid X < \frac{1}{2}) = \frac{69}{128}. \)

(e) \( \mathbb{E}[X] = \int x f_X(x) \, dx = \frac{6}{7} \int_0^{\frac{1}{2}} \left( 2x^3 + x^2 \right) \, dx = \frac{5}{7}. \)

(f) \( \mathbb{E}[Y] = \int \int y f_{X,Y}(x,y) \, dx \, dy = \frac{6}{7} \int_0^1 \left( \int_0^x \left( x^2 y + \frac{xy^2}{2} \right) \, dx \right) \, dy = \frac{6}{7} \int_0^1 \left( 2x^2 + \frac{4x}{3} \right) \, dx = \frac{8}{7}. \)

Problem #10, p. 291:

(a) By symmetry, \( \mathbb{P}(X < Y) = \mathbb{P}(Y < X) \). In addition, \( \mathbb{P}(X = Y) = 0 \), and so \( \mathbb{P}(X < Y) + \mathbb{P}(Y < X) = 1 \). Thus, \( \mathbb{P}(X < Y) = \frac{1}{2} \).

(b) Because the density function for the distribution of \( X \) is \( \int_0^\infty e^{-x-y} \, dy = e^{-x} \) or 0 depending on whether \( X > 0 \) or \( x \leq 0 \), \( \mathbb{P}(X < a) = 1 - e^{-a} \) if \( a > 0 \) and \( \mathbb{P}(X < a) = 0 \) if \( a \leq 0 \).
**Problem #17, p. 292**: First note that, because $P(X_i = X_j) = 0$ when $i \neq j$, the probability being asked about is

$P(X_1 < X_2 < X_3) + P(X_3 < X_2 < X_1)$. Further, observe that the joint distribution of $(X_1, X_2, X_3)$ is the same as the joint distribution of $(X_{i_1}, X_{i_2}, X_{i_3})$, where $(i_1, i_2, i_3)$ is any permutation of $(1, 2, 3)$. Thus, the required probability is $2P(X_{i_1} < X_{i_2} < X_{i_3})$ for any $(i_1, i_2, i_3)$. Finally, again because $P(X_i = X_j) = 0$ when $i \neq j$, the sum over all permutations $(i_1, i_2, i_3)$ of $P(X_{i_1} < X_{i_2} < X_{i_3})$ must be 1, and so $P(X_1 < X_2 < X_3) = \frac{1}{6}$ and the required probability is $\frac{1}{3}$, which was more or less obvious from the outset.

**Problem #27, p. 293**: 

(a) Working with densities:

$$f_X * f_Y(z) = \int_0^\infty 1_{(0,1)}(z-y)e^{-y} \, dy = \int_{(z-1)^+}^z e^{-y} \, dy = \begin{cases} 0 & \text{if } z < 0 \\ 1 - e^{-z} & \text{if } 0 \leq z \leq 1 \\ e^{-z}(e-1) & \text{if } z \geq 1 \end{cases}.$$  

Hence, the distribution $F_Z$ is given by

$$F_Z(z) = \int_{-\infty}^z f_X * f_Y(\xi) \, d\xi = \begin{cases} 0 & \text{if } z < 0 \\ 1 + z - e^{-z} & \text{if } 0 \leq z \leq 1 \\ 1 - (e-1)e^{-z} & \text{if } z \geq 1. \end{cases}$$

(b) Again $F_Z(z) = 0$ when $z \leq 0$, and now

$$F_Z(z) = \int_0^z P(Y \leq zy) \, dy = z \int_0^{\frac{1}{z}} ye^{-y} \, dy + \int_{\frac{1}{z}}^\infty e^{-y} \, dy = z(1 - e^{-\frac{1}{z}}).$$