Solution Set #4, 18.440

Problem #3, p. 171: Let $D_i$ be number on the $i$th die. I originally thought that the problem was to compute $\mathbb{E}[X]$ where $X = D_1 + D_2 + D_4$. Since $\mathbb{P}(D_i = k) = \frac{1}{6}$ for $1 \leq k \leq 6$, $\mathbb{E}[D_i] = 2.5$, and therefore the answer would have been $\mathbb{E}[X] = 3\mathbb{E}[D_i] = 7.5$. However, the problem really was to compute $\mathbb{P}(X = m)$ for $3 \leq m \leq 18$. I know of no easy way to do this. Here is one approach. When $3 \leq m \leq 8$, the number of ways that the 3 die can come up so that their faces add to $m$ is the same as the number $N_m$ of choices of triples $(k_1, k_2, k_3)$ of positive integers such that $k_1 + k_2 + k_3 = m$. There is a one to one correspondence between such triples and the ways to place 2 dividers between members in a row of $m$ positions. Thus, $N_m = \binom{m-1}{2} = \frac{(m-1)(m-2)}{2}$, and so $\mathbb{P}(X = m) = \frac{(m-1)(m-2)}{432}$ when $3 \leq m \leq 8$. When $13 \leq m \leq 18$, notice that

$$\mathbb{P}(X = m) = \mathbb{P}((7 - D_1) + (7 - D_2) + (7 - D_3) = 21 - m) = \mathbb{P}(X = 21 - m) = \frac{(20 - m)(19 - m)}{432}.$$ Finally, when $9 \leq m \leq 12$,

$$\mathbb{P}(X = m) = \frac{1}{6} \sum_{\ell = 1}^{6} \mathbb{P}(D_1 + D_2 = m - \ell) = \frac{1}{6} \sum_{k = m - 6}^{m-1} \mathbb{P}(D_1 + D_2 = k),$$

and

$$\mathbb{P}(D_1 + D_2 = k) = \frac{\min\{(k - 1), (13 - k)\}}{36}$$ for $1 \leq k \leq 12$.

Hence, for $9 \leq m \leq 12$,

$$\mathbb{P}(X = m) = \frac{1}{216} \left( \frac{6 \times 7}{2} - \frac{(m - 8)(m - 7)}{2} + \frac{5 \times 6}{2} - \frac{(13 - m)(14 - m)}{2} \right) = \frac{21m - m^2 - 83}{216}.$$  

Problem #13, p. 172: Let $X_i$ be the money made on sale $i$. Then $\mathbb{P}(X_1 = 1000) = \frac{3}{10} \times \frac{1}{2} = \frac{3}{20}$, $\mathbb{P}(X_2 = 1000) = \frac{9}{10} \times \frac{1}{2} = \frac{9}{20}$, $\mathbb{P}(X_1 + X_2 = 0) = \frac{7}{10}$, and $\mathbb{P}(X_2 = 0) = \frac{1}{10}$. Since $\mathbb{P}(X_1 = a_1 \& X_2 = a_2) = \mathbb{P}(X_1 = a_1)\mathbb{P}(X_2 = a_2)$,

$$\mathbb{P}(X_1 + X_2 = 2000) = \mathbb{P}(X_1 + X_2 = 0)\mathbb{P}(X_2 = 1000),$$

$$\mathbb{P}(X_1 + X_2 = 15 \times 1000) = \mathbb{P}(X_1 = 1000)\mathbb{P}(X_2 = 500) + \mathbb{P}(X_1 = 500)\mathbb{P}(X_2 = 1000),$$

$$\mathbb{P}(X_1 + X_2 = 500) = \mathbb{P}(X_1 = 500)\mathbb{P}(X_2 = 0) + \mathbb{P}(X_1 = 0)\mathbb{P}(X_2 = 500),$$

$$\mathbb{P}(X_1 + X_2 = 0) = \mathbb{P}(X_1 = 0)\mathbb{P}(X_2 = 0).$$

Problem #17, p. 172: $F(x^-) = \lim_{y \to x^-} F(y)$ is the left-limit of $F$ at $x$. 

(a) $\mathbb{P}(X = 1) = F(1) - F(0-) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$, $\mathbb{P}(X = 2) = F(2) - F(1-) = \frac{11}{12} - \frac{3}{4} = \frac{1}{6}$, and $\mathbb{P}(X = 3) = F(3) - F(2-) = \frac{11}{12}$. 

(b) $\mathbb{P}(\frac{1}{2} < X < \frac{3}{2}) = F(\frac{3}{2}) - F(\frac{1}{2}) = \frac{1}{2}.$

Problem #20, p. 173: Let $W$ be the event that she won on the first spin. Then $\mathbb{P}(W) = \frac{18}{38}$, $\mathbb{P}(X = 1|W) = 1$, $\mathbb{P}(X = -3|W) = (\frac{18}{38})^2$, $\mathbb{P}(X = -1|W) = 2 \frac{18}{38} \frac{20}{38}$, and $\mathbb{P}(X = 1) = (\frac{18}{38})^2$. 

(a) $\mathbb{P}(X > 0) = \mathbb{P}(X = 1) = \mathbb{P}(X = 1|W)\mathbb{P}(W) + \mathbb{P}(X = 1|W)\mathbb{P}(W) = \frac{18}{38} + \frac{18^2}{38} = \frac{9 \times 19^2 + 810}{19^3}.$ 

(b) No, because of (c). 

(c) $\mathbb{E}[X] = 1\mathbb{P}(X = 1) - 1\mathbb{P}(X = -1) - 3\mathbb{P}(X = -3) = \frac{9 \times 19^2 + 810 - 1800 - 3000}{19^3} < 0.$

Problem #22, p. 173: Let $N_i$ be the number of games until one team has won $i$ games, and set $q = 1 - p$. Then $\mathbb{E}[N_2] = 2 = q^2 + q^2 = 1 - 2pq$, $\mathbb{E}[N_2] = 3 = \binom{3}{1} (p^2q + pq^2) = 2pq$, and so $\mathbb{E}[N_2] = 2 - 4pq + 6pq = 2 + 3pq \leq 2 + \frac{3}{2}$. Similarly, $\mathbb{P}(N_3 = 3) = p^3 + q^3 = p^2 - pq + q^2 = 1 - 3pq$, $\mathbb{P}(N_3 = 4) = \binom{4}{2} (p^3q + pq^3) = 3pq(1 - 2pq)$, $\mathbb{P}(N_3 = 5) = \binom{4}{1} (p^3q^2 + pq^2) = 6pq^2$, and so $\mathbb{E}[N_3] = 3 - 9pq + 12pq - 24p^2q^2 + 30p^2q^2 = 3 + 3pq + 6pq^2 \leq 3 + \frac{3}{2} + \frac{3}{8}.$