Solution Set # 5

(4.1.9) Let $\Sigma$ be the collection of subsets of the form \( \{ \omega : (X_{i_1}(\omega), \ldots, X_{i_m}(\omega), \ldots) \in \Gamma \} \), where \( \{i_m : m \geq 1\} \subseteq I \) and $\Gamma \in \mathcal{B}^{2^+}$. Clearly $\Sigma$ is a $\sigma$-algebra. Moreover, because

\[
\left\{ \omega : (X_{i_1}(\omega), \ldots, X_{i_m}(\omega), \ldots) \in \prod_{m=1}^{\infty} \Gamma_m \right\} = \bigcap_{m=1}^{\infty} \{ \omega : X_{i_m}(\omega) \in \Gamma_m \} \in \mathcal{F}
\]

for any $\{\Gamma_m : m \geq 1\} \subseteq \mathcal{B}$ and $\mathcal{B}^{2^+}$ is generated by sets of the form $\prod_{m=1}^{\infty} \Gamma_m$, it follows that $\Sigma \subseteq \mathcal{F}$. Conversely, for every $i \in I$ and $\Gamma \in \mathcal{B}$, \( \{ \omega : X_i(\omega) \in \Gamma \} \in \Sigma \), and so $\mathcal{F} \subseteq \Sigma$.

Next, from the preceding, we know that, for any $\{i_m : m \geq 1\} \subseteq I$, $\omega \mapsto (X_{i_1}(\omega), \ldots, X_{i_m}(\omega), \ldots)$ is $\mathcal{F}$-measurable into $(\mathbb{E}^{2^+}, \mathcal{B}^{2^+})$. Hence, if $f : \mathbb{E}^{2^+} \to \mathbb{R}$ is $\mathcal{B}^{2^+}$-measurable, then the function $\omega \mapsto f(X_{i_1}(\omega), \ldots, X_{i_m}(\omega), \ldots)$ is $\mathcal{F}$-measurable. Conversely, if $A \in \mathcal{F}$, then, by the preceding, $1_A(\omega) = 1[(X_{i_1}(\omega), \ldots, X_{i_m}(\omega), \ldots)]$ for some $\{i_m : m \geq 1\} \subseteq I$ and $\Gamma \in \mathcal{B}^{2^+}$. Thus, every $\mathcal{F}$-measurable, simple function has the required form, and therefore, since every $\mathcal{F}$-measurable function is the pointwise limit of simple ones, every $\mathcal{F}$-measurable functions does also.

(4.1.12) Just follow the hint.

(4.2.19) Because $N(n) = \sum_{m=1}^{n}(N(m) - N(m - 1))$ and $\{N(m) - N(m - 1) : m \geq 1\}$ is a sequence of mutually independent, identically distributed random integrable variables with mean value $1$, the strong law implies that $\lim_{n \to \infty} \frac{N(n)}{n} = 1$ (a.s., $\mathbb{P}$). Next, because $t \mapsto N(t)$ is non-decreasing, $\sup_{t \in [n, n+1]} \frac{N(t) - N(n)}{t} \leq \frac{N(n+1) - N(n)}{n}$, and therefore

\[
\mathbb{P}\left( \sup_{t \in [n, n+1]} \frac{N(t) - N(n)}{t} \geq \epsilon \right) \leq \mathbb{P}\left( \frac{N(n+1) - N(n)}{n} \geq \epsilon \right) = \mathbb{P}(N(1) \geq n\epsilon) \leq \frac{\mathbb{E}[N(1)^2]}{n^2 \epsilon^2} = \frac{2}{n^2 \epsilon^2}.
\]

Hence, $\mathbb{P}$-almost surely, $\sup_{t \in [n, n+1]} \frac{N(t) - N(n)}{t} \leq \epsilon$ for all sufficiently large $n$’s, and so, $\mathbb{P}$-almost surely,

\[
\lim_{t \to \infty} \frac{N(t)}{t} - \frac{N([t])}{[t]} \leq \epsilon + \lim_{t \to \infty} \frac{N([t])}{[t]} = \epsilon.
\]

(4.2.20) If $t \mapsto Z(t)$ is $\mathbb{P}$-almost surely non-decreasing, then $Z(1) \geq 0$ (a.s., $\mathbb{P}$) and therefore, by Exercise 3.2.25, $M \in \mathfrak{M}_1(\mathbb{R}^N)$, $M((-\infty, 0)) = 0$, and $m \geq \int_{[-1,1]} y M(dy)$. Conversely, if $M \in \mathfrak{M}_1(\mathbb{R}^N)$, $M((-\infty, 0)) = 0$, and $m \geq \int_{[-1,1]} y M(dy)$, then, by Exercise 3.2.25, for each $0 \leq s \leq t < \infty$, $Z(t) - Z(s) \geq 0$ (a.s., $\mathbb{P}$). Hence, by right continuity, $\mathbb{P}$-almost surely, $Z(t) - Z(s) \geq 0$ for all $0 \leq s \leq t < \infty$.

(4.2.23) The reduction, described in (iv), to the case when $\nu(\{0\}) = 0$ is easy. Thus, assume that $\nu(\{0\}) = 0$ throughout. Then, $M(\{0\}) = \nu(\{0\}) = 0$, and, for each $r > 0$, $M(B(0,r)) = \nu(B(0,r)) < \infty$. Hence, $M \in \mathfrak{M}_\infty(\mathbb{R}^N)$. Next, if $\{\Gamma_m : 1 \leq m \leq n\}$ are mutually disjoint Borel sets, then so are $\{F(\Gamma_m) : 1 \leq m \leq n\}$, and therefore $P\{\Gamma_1, \cdot, \cdot\} = j(1,F(\Gamma)), \ldots, P\{\Gamma_n, \cdot, \cdot\} = j(1,F(\Gamma)), \cdot$ are mutually independent random variables. Finally, for any $\Gamma \in \mathcal{B}^\infty$, $P(\Gamma, \cdot) = j(1,F(\Gamma), \cdot)$ is a Poisson random variable with rate $M(F(\Gamma)) = \nu(\Gamma)$. 