Solution Set # 3

(2.1.11) Suppose that $\mu \in \mathcal{P}$, and let $\{X_n : n \geq 1\}$ be a sequence of independent random variables with distribution $\mu$. Then $T_2^\sigma \mu$ is the distribution of $\tilde{S}_{2n}$, and so, by the Central Limit Theorem, $T_2^\sigma \mu \Rightarrow \gamma_{0,1}$. In particular, if $\mu = T_2^\sigma \mu$ and therefore $\mu = T_2^\sigma \mu$ for all $n \geq 1$, then $\mu = \gamma_{0,1}$.

(2.1.13) (i) Let $\lambda_{S^{n-1}}$ denote surface measure on $S^{n-1}$. Using cylindrical coordinates, one can show (cf. Exercise 5.2.21 in my Essential of Integration Theory for Analysis) that, for $n \geq 2$,

$$
\lambda_{S^{n-1}}(d\theta) = (1 - \theta^2_1)^{n-3} \lambda_{S^{n-2}}(d\theta') \lambda_{(-1,1)}(d\theta_n),
$$

where $\lambda_{(-1,1)}$ is Lebesgue measure on $(-1,1)$ and $\theta' = (\theta_2, \ldots, \theta_n)$. Hence, because

$$
\int_{S^{n-1}(r)} \varphi d\lambda_{S^{n-1}(r)} = r^{n-1} \int_{S^{n-1}} \varphi(r\theta) \lambda_{S^{n-1}}(d\theta),
$$

we obtain the asserted expression for $\lambda_n^{(1)}$, and, given that expression, the rest is easy.

(ii) Simply follow the outline.

(iii) First reduce to the case when $(\varphi, \gamma_{0,1}) = 0$. In that case, note that, for any $1 \leq k \neq \ell \leq n$, the distribution of $(x_k, x_\ell)$ under $\lambda_n$ is $\lambda_n^{(2)}$, and therefore

$$
\int \left( \frac{1}{n} \sum_{k=1}^n \varphi(x_k) \right)^2 \lambda_n(dx) = \frac{1}{n} \int \varphi(x_1)^2 \lambda_n(dx) + \frac{n-1}{n} \int \varphi(x_1) \varphi(x_2) \lambda_n(dx) \rightarrow \langle \varphi, \gamma_{0,1} \rangle^2 = 0.
$$

(2.3.21) For parts (i) and (ii), just follow the outline. To do part (iii), for each $e \in \mathbb{R}^N$, let $\mu_e$ be the distribution of $x \sim (e, x)_{\mathbb{R}^N}$ under $\mu$, apply (ii) to $\mu_e$, and conclude that $\mu_e(\xi) = e^{-\frac{\pi^2}{2\sigma^2}}$, where $\sigma^2_e = \int (e, x)^2_{\mathbb{R}^N} \mu(dx)$. Thus, $\mu \in \mathcal{N}(0, C)$, where $C$ is determined by $(\xi, C\eta)_{\mathbb{R}^N} = \int (\xi, x)_{\mathbb{R}^N} \eta(x)_{\mathbb{R}^N} \mu(dx)$.

(2.3.23) If $\mu = T_n \mu$ for some $\mu$ on $\mathbb{R}$, then $\mu^N = T_n \mu^N$, and so there exists a non-trivial solution on $\mathbb{R}^N$ if there exists one on $\mathbb{R}$. Conversely, if $\mu$ on $\mathbb{R}^N$ is a non-trivial solution, then there exists a $e \in S^{N-1}$ such that the distribution $\mu_e$ of $x \in \mathbb{R}^N \mapsto (e, x)_{\mathbb{R}^N} \in \mathbb{R}$ under $\mu$ is non-trivial, and $\mu_e$ is a solution for all $e \in S^{N-1}$. Thus, the question of existence reduces to the case when $N = 1$, in which case the outline shows how to solve it.

(2.3.25) If $\sigma = 0$ there is nothing to do, and $\frac{X}{\sigma} \in \mathcal{N}(0, 1)$ when $\sigma > 0$. Hence, we may assume that $\sigma = 1$. Next, when $\sigma = 1$,

$$
\mathbb{E}^P[|X|^p] = \sqrt{\frac{2}{\pi}} \int_0^\infty x^p e^{-\frac{x^2}{2}} dx
$$

and

$$
(*) \int_0^\infty x^p e^{-\frac{x^2}{2}} dx = 2^{\frac{p+1}{2}} \int_0^\infty t^{\frac{p-1}{2}} e^{-t} dt = 2^{\frac{p-1}{2}} \Gamma\left(\frac{p+1}{2}\right).
$$

Hence, (2.3.26) follows.
(i) First note that $\frac{X}{\beta}$ is 1-sub-Gaussian, and so it is enough to handle the case when $\beta = 1$. In this case, the first estimate in Lemma 2.3.18 says that $\mathbb{P}(X \geq R) \leq 2e^{-\frac{2\sigma^2}{R^2}}$, and so, by (*),

$$E^p[|X|^p] = p \int_0^\infty t^{p-1} \mathbb{P}(|X| > t) \, dt \leq 2p \int_0^\infty t^{p-1} e^{-\frac{t^2}{2\sigma^2}} \, dt = p2\sigma^p \Gamma\left(\frac{p}{2}\right).$$

(ii) Again, by considering $\frac{X}{\beta}$, one can reduce to the case when $\beta = 1$ after replacing $\sigma$ by $\frac{\sigma}{\beta}$. Thus, assume that $\beta = 1$. By Hölder’s inequality, $E^p[|X|^p] \geq \sigma^p$ for $p \geq 2$. Now assume that $p < 2$. If $q \in (1, \infty)$, then $2 = \frac{2}{q} + \frac{1}{q'}$. Thus, by Hölder’s inequality,

$$\sigma^2 = E^p[|X|^\frac{p}{2} |X|^{\frac{p}{2} - 1}] \leq E^p[|X|^p]^{\frac{1}{2}} E^p[|X|^{\frac{2q-p}{2q-1}}]^{\frac{1}{2}}.$$

Now choose $q = \frac{4-p}{2}$. Then $\frac{2q-p}{2q-1} = 4$, and the result follows by simple arithmetic.

(iii) Because $S$ is $B$-sub-Gaussian and has variance $\Sigma^2$, there is nothing to do.

(iv) Follow the outline.

(2.2.28) and (2.3.29) Follow the outline.