(6.2.8):

(i) When \( q = \infty \) or \( \mu(E) = 0 \), there is nothing to do. In addition, by the Monotone Convergence Theorem, it suffices to treat bounded \( f \)'s. Thus, assume that \( q < \infty \), \( 0 < \mu(E) < \infty \), and \( f \) is bounded. Note that by replacing \( \mu \) by \( \frac{\mu}{\mu(E)} \), one can reduce to the case where \( \mu(E) = 1 \). But when \( \mu(E) = 1 \), Jensen’s Inequality implies that

\[
\left( \int |f|^q \, d\mu \right)^{\frac{1}{q}} \geq \int |f|^p \, d\mu,
\]

since \( x \in [0, \infty) \mapsto x^{\frac{p}{q}} \in [0, \infty) \) is a continuous, concave function.

(ii) First observe that \( |f(x)|^p \leq \int |f|^p \, d\mu \) for all \( x \in E \) and \( p \in [0, \infty) \). Thus, we need only worry about \( p \in [1, \infty) \). Following the given hint, let \( \{x_1, \ldots, x_n, \ldots\} \) be an enumeration of \( E \), and observe that, for any \( p \in [1, \infty) \),

\[
\|f\|_{L^p(\mu; \mathbb{R})} = \left( \sum_{m=1}^{\infty} a_m^p \right)^{\frac{1}{p}} = \lim_{n \to \infty} \left( \sum_{m=1}^{n} a_m^p \right)^{\frac{1}{p}},
\]

where \( a_m = |f(x_m)| \). Thus, it suffices to treat the case when \( E = \{1, \ldots, n\} \) and \( \mu(\{m\}) = 1 \) for \( 1 \leq m \leq n \), in which case the required result is equivalent to showing that \( (\sum_{m=1}^{n} a_m^p)^{\frac{1}{p}} \) is a non-decreasing function of \( p \in [1, \infty) \) for all \( \{a_1, \ldots, a_m\} \subseteq [0, \infty) \). To prove this, set \( A_p = \sum_{m=1}^{n} a_m^p \). If \( A_p = 0 \) for some, and therefore all, \( p \in [1, \infty) \), then there is nothing to do. Thus, assume that \( A_p > 0 \) for all \( p \in [1, \infty) \). If one knew that \( A_1 \leq 1 \implies A_p \leq 1 \) for all \( p \in [1, \infty) \), then one would know that

\[
A_p^{-\frac{2}{p}} A_q = \sum_{m=1}^{n} \left( \frac{a_m^p}{A_p} \right) \leq 1
\]

and therefore that \( A_q^\frac{1}{p} \leq A_p^\frac{1}{p} \) for \( 1 \leq p < q < \infty \). Finally, showing that \( A_1 \leq 1 \implies A_p \leq 1 \) is trivial, since, for \( a \in [0, 1], p \in [1, \infty) \mapsto a^p \in [0, \infty) \) is a non-decreasing function.

(iii) First suppose that \( \mu(E) < \infty \). By (i), we then know that

\[
\lim_{p \to \infty} \|f\|_{L^p(\mu; \mathbb{R})} \leq \|f\|_{L^\infty(\mu; \mathbb{R})}.
\]

Next, suppose that \( a < \|f\|_{L^\infty(\mu; \mathbb{R})} \), and set \( \Gamma = \{|f| > a\} \). Then \( \mu(\Gamma) \in (0, \infty) \) and \( \|f\|_{L^p(\mu; \mathbb{R})} \geq a \mu(\Gamma)^{\frac{1}{p}} \), which means that \( \lim_{p \to \infty} \|f\|_{L^p(\mu; \mathbb{R})} \geq a \).

To treat the case when \( \mu(E) = \infty \) and \( f \in L^1(\mu; \mathbb{R}) \), define \( \nu(\Gamma) = \int_{\Gamma} |f| \, d\mu \), note that \( \nu(E) < \infty \), that \( \|f\|_{L^\infty(\mu; \mathbb{R})} = \|f\|_{L^\infty(\nu; \mathbb{R})} \), and that \( \|f\|_{L^p(\mu; \mathbb{R})} = \|f\|_{L^{p-1}(\nu; \mathbb{R})} \) for \( p \in [2, \infty) \).

(iv) There is nothing to do.
(6.2.11): Follow the outline, and use the hint for part (iii).

(6.3.18):
(i) By definition,
\[ \gamma_{\sqrt{s}} * \gamma_{\sqrt{t}}(x) = \frac{1}{(2\pi \sqrt{st})^N} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{2s}} e^{-\frac{|y|^2}{2t}} \, dy. \]

By elementary algebra,
\[ \frac{|x-y|^2}{s} + \frac{|y|^2}{t} = \frac{s+t}{st} |x|^2 + \frac{1}{s+t} |x|^2. \]

Thus,
\[ \gamma_{\sqrt{s}} * \gamma_{\sqrt{t}}(x) = e^{-\frac{|x|^2}{2(s+t)}} \int_{\mathbb{R}^N} e^{-\frac{(s+t)|y|^2}{2st}} \, dy = \gamma_{\sqrt{s+t}}(x), \]
where, in the final steps I have used the translation invariance of \( \lambda_{\mathbb{R}^N} \) and (i) from Exercise 5.1.13.

(ii) Take \( \Phi(\xi) = \frac{\xi}{\eta - \xi} \) as suggested in the hint. Then
\[ \frac{1}{\xi} = \frac{1 + \Phi(\xi)}{\eta \Phi(\xi)}, \quad \frac{1}{\eta - \xi} = \frac{1 + \Phi(\xi)}{\eta}, \quad \text{and} \quad \phi'(\xi) = \frac{\eta}{(\eta - \xi)^2} = \frac{(1 + \Phi(\xi))^2}{\eta}, \]
and so
\[ \int_{(0,\eta)} (\xi(\eta - \xi))^{-\frac{1}{2}} e^{-\frac{\xi^2}{2\eta^2}} e^{-\frac{\xi}{\eta}} d\xi \]
\[ = \frac{e^{-\frac{\xi^2}{2\eta^2}}}{\eta} \int_{(0,\eta)} (\Phi(\xi))^{-\frac{1}{2}} + \Phi(\xi)^{-\frac{1}{2}} \cdot e^{-\frac{\xi^2}{2\eta^2}} \cdot \frac{\eta^2}{\eta^2} \Phi'(\xi) d\xi \]
\[ = \frac{e^{-\frac{\xi^2}{2\eta^2}}}{\eta^2} \int_{(0,\infty)} (\xi^{-\frac{1}{2}} + \xi^{-\frac{1}{2}}) \cdot e^{-\frac{\xi^2}{2\eta^2}} \cdot \frac{\eta^2}{\eta^2} d\xi = \frac{\pi^{\frac{1}{2}}(s+t) e^{-\frac{(s+t)^2}{2}}} {st\eta^\frac{3}{2}}, \]
where I have Exercise 5.1.10 and (iv) from Exercise 5.1.13.

(iii) To prove the first equality, note that
\[ \int_{(0,\infty)} \gamma_{\sqrt{s}}(x) \nu_\xi(\xi) \, d\xi = \frac{t}{\pi^{\frac{N+1}{2}}} \int_{(0,\infty)} \xi^{-\frac{N+1}{2}} e^{-\frac{\xi^2 + x^2}{t}} \, d\xi \]
\[ = \frac{t}{(\pi(|x|^2 + t^2))^{\frac{N+1}{2}}} \int_{(0,\infty)} \eta^{-\frac{N+1}{2}} e^{-\eta} \, d\eta = \frac{t}{(\pi(|x|^2 + t^2))^{\frac{N+1}{2}}} \Gamma\left(\frac{N+1}{2}\right). \]
Hence, since $\frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{1/2}} = \frac{2}{\omega_n}$, we are done. Given the first equality and the results in (i) and (ii), one has

$$P_s \ast P_t(x) = \int \int \gamma \sqrt{\frac{a}{2^6}}(x)\nu_2(\xi)\nu_2(\eta) d\xi d\eta$$

$$= \int \gamma \sqrt{\frac{a}{2^6}}(x) \left( \int \nu_2(\eta - \xi)\mu_2(\xi) d\xi \right) d\eta$$

$$= \int \gamma \sqrt{\frac{a}{2^6}}(x)\nu_{s+t}^2(\eta) d\eta = P_{s+t}(x).$$

(iv) Clearly

$$g_\alpha \ast g_\beta(x) = \frac{e^{-x}}{\Gamma(\alpha)\Gamma(\beta)} \int_0(x-y)'^{\alpha-1}y'^{\beta-1} dy$$

$$= \frac{x'^{\alpha+\beta-1}e^{-x}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 (1-t)^{\alpha-1}t^{\beta-1} dt = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} B(\alpha, \beta)g_{\alpha+\beta}(x).$$

Thus, since both $g_\alpha \ast g_\beta$ and $g_{\alpha\beta}$ have integral 1, this proves both that $B(\alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$ and that $g_\alpha \ast g_\beta = g_{\alpha+\beta}$.

(6.3.20): Just follow the outline and use the hint.

(7.1.11): Suppose that $\Pi$ is the orthogonal projection onto $L$. Then it is obvious that $L = \text{Range}(\Pi)$ and that $\Pi^2 = \Pi$. Furthermore, because $x - \Pi x \perp L$, $x, \Pi y)_H = (\Pi x, y)_H = (\Pi x, y)_H$, and so $\Pi$ is symmetric. Now assume that $L = \text{Range}(\Pi)$, $\Pi^2 = \Pi$, and $\Pi$ is symmetric. Given $y \in L$, $(x - \Pi x, y)_H = (x - \Pi x, \Pi y)_H = (\Pi x - \Pi^2 x, y)_H = 0$, and so $\Pi$ is orthogonal projection onto $L$.

(7.1.13): It suffices to show that if $E'$ is an orthonormal basis for $L^\perp$, then $E \cup E'$ is an orthonormal basis for $H$. But it is obvious that $E \cup E'$ is orthonormal. In addition, if $x \in H$, then $\Pi_L x = \sum_{e \in E^\perp(x, e)} H e$ and, by Exercise 7.1.11, $(I - \Pi_L) x = \Pi_{L^\perp} x = \sum_{e' \in E^\perp(x, e')} H e'$, where, in both cases, the convergence is in $H$. Hence, $x = \Pi_L x + (I - \Pi_L) x = \sum_{e \in E \cup E'}(x, e)' H e$, and so $E \cup E'$ is an orthonormal basis for $H$.

(7.2.12): Only part (iii) requires comment. Following the hint, consider the extension $\tilde{f}$ of $f \in L^2(\lambda_{[0,1]}; \mathbb{C})$ to $[-1,1]$ as an even function. Then

$$\left( f, 1 \right)_{L^2(\lambda_{[0,1]}; \mathbb{C})} = 2^{-1} \left( \tilde{f}, 1 \right)_{L^2([-1,1]; \mathbb{C})} = \left( \tilde{f}, e_{[-1,1],0} \right)_{L^2([-1,1]; \mathbb{C})} e_{[-1,1],0},$$

and, for $n \geq 1$,

$$\left( f, 2^{\frac{1}{2}} \cos(\pi n \cdot) \right)_{L^2(\lambda_{[0,1]}; \mathbb{C})} \left( 2^{\frac{1}{2}} \cos(\pi n \cdot) \right) = \left( \tilde{f}, e_{[-1,1,n]} \right)_{L^2([-1,1]; \mathbb{C})} + \left( \tilde{f}, e_{[-1,1],-n} \right)_{L^2([-1,1]; \mathbb{C})}.$$
(7.2.13): Just follow the outline given.

(7.2.14): Because \( \sum_{n \in \mathbb{Z}} \left| (\varphi, e_n)_{L^2(\lambda_{[0,1]} ; C)} \right| < \infty \), it is obvious that the series \( \sum_{n \in \mathbb{Z}} (\varphi, e_n)_{L^2(\lambda_{[0,1]} ; C)} e_n \) converges uniformly on \([0, 1]\) to a continuous function. At the same time, we know that it converges in \( L^2(\lambda_{[0,1]} ; C) \) to \( \varphi \). Hence, the continuous function to which it converges uniformly is Lebesgue-almost everywhere equal to \( \varphi \), which, since \( \varphi \) is also continuous, means that it must be converging uniformly to \( \varphi \).