(5.2.20):

(i) Set \( F(s, t) = s^{\alpha-1}t^{\beta-1}e^{s+t} \) and \( \Phi(u, v) = \left( \frac{uv}{u(1 - v)} \right) \). Then \( \Phi \) is a diffeomorphism from \((0, \infty) \times (0, 1)\) onto \((0, \infty)^2\), and \( J\Phi(u, v) = u \). Hence,

\[
\Gamma(\alpha)\Gamma(\beta) = \int \int_{(0, \infty)^2} F(s, t) \, ds \, dt = \int \int_{(0, \infty) \times (0, 1)} F \circ \Phi(u, v) J\Phi(u, v) \, du \, dv
\]

\[
= \int \int_{(0, \infty) \times (0, 1)} u^{\alpha+\beta-1}e^{-u-v^{\alpha-1}(1 - v)^{\beta-1}} \, du \, dv = \Gamma(\alpha + \beta)B(\alpha, \beta).
\]

(ii) Clearly, by Theorem 5.1.8,

\[
\int_{\mathbb{R}^N} \frac{1}{(1 + |x|^2)^{\lambda}} \, dx = \omega_{N-1} \int_{(0, \infty)} r^{N-1}(1 + r^2)^{-\lambda} \, dr.
\]

Next, take \( \Phi(r) = \frac{r^2}{1 + r^2} \). Then \( \Phi \) is a diffeomorphism from \((0, \infty) \times (0, 1)\) and \( J\Phi(r) = \frac{2r}{1 + r^2} \). Thus

\[
\int_{(0, \infty)} r^{N-1}(1 + r^2)^{-\lambda} \, dr = \frac{1}{2} \int_{(0, \infty)} \Phi(r)^\frac{N}{2} - 1 \, (1 - \Phi(r))^{-\frac{\lambda}{2} - 1} J\Phi(r) \, dr
\]

\[
= \frac{1}{2} \int_{(0, 1)} t^{\frac{\lambda}{2} - 1}(1 - t)^{\lambda - \frac{N}{2} - 1} \, dt = \frac{1}{2} B\left( \frac{\lambda}{2}, \lambda - \frac{N}{2} \right).
\]

(iii) First note that

\[
\int_{(-1, 1)} (1 - \xi^2)^{\frac{N}{2} - 1} \, d\xi = 2 \int_{(0, 1)} (1 - \xi^2)^{\frac{N}{2} - 1} \, d\xi.
\]

Next, check that \( \Phi(\xi) = \xi^2 \) is a diffeomorphism from \((0, 1) \times (0, 1)\) onto itself and that \( J\Phi(\xi) = 2\xi \). Thus,

\[
\int_{(0, 1)} (1 + \xi^2)^{\frac{N}{2} - 1} \, d\xi = \frac{1}{2} \int_{(0, 1)} \Phi(\xi)^{-\frac{1}{2}} (1 - \Phi(\xi))^{\frac{N}{2} - 1} J\Phi(\xi) \, d\xi
\]

\[
= \frac{1}{2} \int_{(0, 1)} t^{-\frac{1}{2}}(1 - t)^{\frac{N}{2} - 1} \, dt = \frac{1}{2} B\left( \frac{1}{2}, \frac{N}{2} - 1 \right).
\]
(5.2.21)

(i) By Fubini’s Theorem and Theorem 5.1.8,
\[
\int_{(0,\infty)} r^N \left( \int_{(-1,1) \times \mathbb{S}^{N-1}} F(r \Xi(\rho, \omega)) \mu_N(d\rho \times d\omega) \right) dr
\]
\[
= \int_{(-1,1)} (1 - \rho^2)^{\frac{N}{2} - 1} \left( \int_{(0,\infty) \times \mathbb{S}^{N-1}} r^N F((1 - \rho^2)^{\frac{3}{2}} r \rho, r \rho) dr \times \lambda_{\mathbb{S}^{N-1}}(d\omega) \right) d\rho
\]
\[
= \int_{(-1,1)} (1 - \rho^2)^{\frac{N}{2} - 1} \left( \int_{\mathbb{R}^N} |x| F((1 - \rho^2)^{\frac{3}{2}} x, |x| \rho) \lambda_{\mathbb{R}^N}(dx) \right) d\rho
\]
\[
= \int_{\mathbb{R}^N \times (-1,1)} |x|(1 - y^2)^{\frac{N}{2}} F \circ \Phi(x, y) \lambda_{\mathbb{R}^N+1}(dx \times dy).
\]

(ii) Clearly \( \Phi \) is one-to-one and onto. In fact, if \((u, v) \in (\mathbb{R}^N \times \mathbb{R}) \setminus \{0, 0\} \), then
\[
\Phi^{-1}(u, v) = \left( |u|^2 + v^2 \right)^{-\frac{1}{2}} \left( \frac{u}{v} \right).
\]

To compute \( J\Phi \), first note that the Jacobian matrix for \( \Phi \) at \((x, y)\) can be expressed as
\[
\begin{pmatrix}
(1 - y^2)^{\frac{3}{2}} I_{\mathbb{R}^N} & -(1 - y^2)^{-\frac{3}{2}} xy^\top \\
\frac{y}{|x|} x & |x|
\end{pmatrix}
\]
where \( x \) is thought of as a row vector and \( x^\top \) is the corresponding column vector. Using column operations, one can eliminate the first \( N \) entries from the \((N + 1)\)st column to see that
\[
\det \begin{pmatrix}
(1 - y^2)^{\frac{3}{2}} I_{\mathbb{R}^N} & -(1 - y^2)^{-\frac{3}{2}} xy^\top \\
\frac{y}{|x|} x & |x|
\end{pmatrix}
\]
\[
= \det \begin{pmatrix}
(1 - y^2)^{\frac{3}{2}} I_{\mathbb{R}^N} & 0 \\
\frac{y}{|x|} x & (|x|(1 - y^2)^{-1})^{-1}
\end{pmatrix}
\]
\[
= |x|(1 - y^2)^{\frac{N}{2} - 1}.
\]

By combining this with (i), we know that
\[
\int_{(0,\infty)} r^N \left( \int_{(-1,1) \times \mathbb{S}^{N-1}} F(r \Xi(\rho, \omega)) \mu_N(d\rho \times d\omega) \right) dr
\]
\[
= \int_{\mathbb{R}^N \setminus \{0\}} F \circ \Phi \ d\lambda_{\mathbb{R}^N+1}.
\]

Now apply Theorems 5.2.2 and 5.1.8 to arrive at
\[
\int_{(0,\infty)} r^N \left( \int_{(-1,1) \times \mathbb{S}^{N-1}} F(r \Xi(\rho, \omega)) \mu_N(d\rho \times d\omega) \right) dr
\]
\[
= \int_{\mathbb{R}^N+1} F \ d\lambda_{\mathbb{R}^N+1}
\]
\[
= \int_{(0,\infty)} r^N \left( \int_{\mathbb{S}^N} F(r \omega) \lambda_{\mathbb{S}^N}(d\omega) \right) dr.
\]
(iii) Using $F$’s of the form suggested, it becomes clear from (ii) that
\[
\int_{\mathbb{S}^N} \psi(\omega) \lambda_{\mathbb{S}^N}(d\omega) = \iint_{(-1,1) \times \mathbb{S}^{N-1}} \psi \circ \Xi(\rho, \omega) \mu_N(d\rho \times d\omega),
\]
which shows that $\lambda_{\mathbb{S}^N} = \Xi_* \mu_N$.

(iv) First use orthogonal orthogonal invariance to see that
\[
\int_{\mathbb{S}^N} f((\theta, \eta)_{\mathbb{R}^{N+1}}) \lambda_{\mathbb{S}^N}(d\eta)
\]
is the same for all $\theta \in \mathbb{S}^N$. Thus, it is suffices to treat the case when $\theta$ is the unit vector in the direction of the $(N+1)$st coordinate, in which case the preceding shows that
\[
\int_{\mathbb{S}^N} f((\theta, \eta)_{\mathbb{R}^{N+1}}) \lambda_{\mathbb{S}^N}(d\eta) = \int_{\mathbb{S}^N} f(\eta_{N+1}) \lambda_{\mathbb{S}^N}(d\eta) = \iint_{(-1,1) \times \mathbb{S}^{N-1}} f(\rho) \mu_N(d\rho \times d\omega) = \omega_{N-1} \int_{(-1,1)} f(\rho)(1 - \rho^2)^{N-1} d\rho.
\]

(5.2.25): To see that $M$ is a hypersurface, define $F : U \times \mathbb{R} \to \mathbb{R}$ by $F(u, y) = y - f(u)$, note that $M = \{(u, y) \in U \times \mathbb{R} : F(u, y) = 0\}$, and check that
\[
\nabla F(u, y) = (-\partial_u f(u), \ldots, -\partial_{u_N} f(u), 1).
\]
Hence, $|\nabla F|$ never vanishes. To show that $(\Psi, U)$ is a global coordinate chart, first note the $\Psi$ is obviously one-to-one from $U$ onto $M$. Second, because $\partial_u \Psi(u) = e_i + \partial_u f(u)e_N$, where $(e_1, \ldots, e_N)$ is the standard, orthonormal basis for $\mathbb{R}^N$, it is clear that $\{\partial_u \Psi(u), \ldots, \partial_{u_N} \Psi(u)\}$ is a linearly independent subset of $\mathbb{R}^N$, and that $(\partial_u \Psi(u), \nabla F(u, 0))_{\mathbb{R}^N} = 0$ for each $1 \leq i \leq N - 1$. Hence, $(\Psi, U)$ is a global coordinate chart for $M$. Further, from the preceding it is easy to see that
\[
\left(\left(\partial_u \Psi, \partial_{u_N} \Psi\right)_{\mathbb{R}^N}\right)_{1 \leq i, j \leq N-1} = I_{\mathbb{R}^{N-1}} + (\nabla f)^\top \nabla f,
\]
and following the hint one sees that $J\Psi = \sqrt{1 + |\nabla f|^2}$.

(5.2.26):

(i) For each $x \in M$ choose $r_x > 0$ so the $B(x, r_x) \subseteq G$ and $\partial_{x_N} F$ never vanishes on $B(x, r_x)$. Then, because $M$ is connected, $H = \bigcup_{x \in M} B(x, r_x)$ is a connected, open neighborhood of $M$, $H \subseteq G$, and $\partial_{x_N} F$ never vanishes on $H$. Thus, without loss in generality, I will assume that $\partial_{x_N} F > 0$ on $H$. Next, if $x, y \in H$ and $\Phi(x) = \Phi(y)$, then $(x_1, \ldots, x_{N-1}) = (y_1, \ldots, y_{N-1})$ and
\[
0 = F(x_1, \ldots, x_{N-1}, y_N) - F(x_1, \ldots, x_{N-1}, x_N) = \int_{[x_N, y_N]} \partial_{x_N} F(x_1, \ldots, x_{N-1}, \xi) d\xi,
\]
which is possible only if $x_N = y_N$. Hence, since $J\Phi = \partial_{x_N} F$, $\Phi$ is a diffeomorphism on $H$, and so, by the Inverse Function Theorem, $f$ exists and has the asserted properties.
(ii) Since
\[ 0 = \partial_u F(u, f(u)) = \partial_{x_i} F(u, f(u)) + \partial_{x_N} F(u, f(u)) \partial_u f(u), \]
we know that
\[ 1 + |\nabla f(u)|^2 = 1 + \sum_{i=1}^{N-1} \left( \frac{\partial_x F(u, f(u))}{\partial_{x_N} F(u, f(u))} \right)^2 = \left( \frac{\nabla F(u, f(u))}{\partial_{x_N} F(u, f(u))} \right)^2. \]
Thus, the desired result follows from Exercise 5.2.25.

(5.3.11): Given the hint, there is nothing more to do.

(5.3.14): The reduction to the case when \( 0 \in G \) and \( \zeta = 0 \) is a simple matter of translation. To see that \( \partial_z \frac{f(z)}{z} = \frac{\partial f(z)}{z} \) for \( z \neq 0 \), apply the chain rule and the fact that \( \frac{1}{z} \) is analytic in \( \mathbb{C} \setminus \{0\} \) and therefore, by the Cauchy–Riemann equation, satisfies \( \partial_z \frac{1}{z} = 0 \). Checking that \( \eta \in C_0(\mathbb{R}^2; [0,1]) \) is elementary calculus, and the rest follows by applying Exercise 5.3.11 to \( f_r \) in the regions \( G \) and \( B(0,r) \).

(6.1.6): The only part that does not follow immediately from the outline is the derivation of Minkowski’s Inequality for \( p = 2 \) from Schwarz’s Inequality. But, by Schwarz’s Inequality,
\[ \int (f_1 + f_2)^2 \, d\mu = \int f_1^2 \, d\mu + 2 \int f_1 f_2 \, d\mu + \int f_2^2 \, d\mu \]
\[ \leq \left( \int f_1^2 \, d\mu \right)^{\frac{1}{2}} \left( \int f_2^2 \, d\mu \right)^{\frac{1}{2}} + \int f_2^2 \, d\mu \]
\[ = \left( \sqrt{\int f_1^2 \, d\mu} + \sqrt{\int f_2^2 \, d\mu} \right)^2, \]
from which Minkowski’s Inequality for \( p = 2 \) is clear.

(6.1.9): Obviously, if there exists an \( e_q \in S^{N-1} \) such that \( (e_q, q - x)_{\mathbb{R}^N} > 0 \) for all \( x \in C \), then \( q \notin C \). Now suppose that \( q \notin C \). Choose \( p \in C \) so that \( |q - p| = \min \{|q - x| : x \in C\} \), and set \( e_q = \frac{q - p}{|q - p|} \). Then, for any \( x \in C \),
\[ 0 \leq \frac{d}{d\theta} |q - \theta x - (1 - \theta)p|_{\theta=0}^2 = 2(q - p, p - x)_{\mathbb{R}^N}, \]
and so \( (e_q, p - x)_{\mathbb{R}^N} \geq 0 \) for all \( x \in C \). Hence
\[ (e_q, q - x)_{\mathbb{R}^N} = (e_q, q - p)_{\mathbb{R}^N} + (e_q, p - x)_{\mathbb{R}^N} \geq |q - p| > 0 \quad \text{for all} \ x \in C. \]

Next, let \( F \) be given, and define \( p \) accordingly. If \( p \notin C \), then there would exist an \( e \in S^{N-1} \) such that \( (e, p - F)_{\mathbb{R}^N} > 0 \). On the other hand,
\[ \int (e, p - F)_{\mathbb{R}^N} \, d\mu = (e, p - \int F \, d\mu)_{\mathbb{R}^N} = 0, \]
and so $p$ must be an element of $C$.

Turning to the proof of Jensen’s Inequality, it is easy to check that $g_1 \wedge g_2$ is a continuous, concave function on $C$ if $g_1$ and $g_2$ are. Thus, if one knows Jensen’s Inequality when $g$ is bounded, then

$$\int g \circ F \, d\mu = \lim_{n \to \infty} \int (g \wedge n) \circ F \, d\mu \leq \lim_{n \to \infty} (g \wedge n) \left( \int F \, d\mu \right) = g \left( \int F \, d\mu \right).$$

Now, assume that $g$ is bounded. Then it is clear that $\hat{C}$ is a closed, convex subset of $\mathbb{R}^{N+1}$ and that $\hat{F}$ is a $\mu$-integrable, $\hat{C}$-valued function. Hence

$$\left( \int F \, d\mu \right) \in \hat{C},$$

which is equivalent to saying that Jensen’s Inequality holds.