18.310 Problem Set #7
Due: Wednesday, November 8, 2006

Problem 1:
State the (Kempe) false proof of the Four Color Theorem in your own words, and show what is wrong with it.

Problem 2:
Read the supplementary notes on Kuratowski’s Theorem and prove the claims in A, B and C at the end of the notes; namely, the statements that the bridge connection sequences given in A and B imply that G is not minimally non-planar and that it possesses a longer separating cycle. Also prove that any connection sequence in which no bridge is connected to adjacent vertices of C either is or contains 123123 or one of the sequences described in A or B or a relabeling and/or cycling of one of these.

Problem 3:
Read the new Matching Problems Lecture notes. Prove: If all vertices of G have the same degree (number of incident edges) and G is bipartite, then there is a complete matching in G (This is a set of pairs such that each pair is an edge of G, and every vertex of G is paired). Further, the edges of G can be partitioned into complete matchings.

Problem 4:
A latin square is an $N$ by $N$ square array of integer each of which is in the range from 1 to $N$ so that no integer appears twice in a row or column. Prove: If you have a $k$ by $N$ array of integers from 1 to $N$ obeying the same condition, you can complete it to a latin square. (Hint: To fill in one additional row with integers, the integers and places in the row form a bipartite graph defined by eligibility of an integer to appear in that place.)

Problem 5:

(A) Find a graph (on 6 vertices) whose coloring number is 2 but whose list coloring number is 3. How many vertices do you need to make the list coloring number of a bipartite graph equal to 4? The coloring number of a graph is the minimum number of colors needed to color all the vertices of a graph so that all edges have two colors on their vertices. The list coloring number $k$ is the maximum needed to achieve the same result when each vertex must be colored by colors on a list of length at least $k$. The maximum is taken over all possible lists.
(B) Show how to find a matching in a bipartite graph having \( N \) vertices in each part, in which each vertex has degree \( d \) obeying \( d = 2^k \), by wandering over a total of \( 2Nd \) edges. (Hint: Find cycles and modify them.)

Problem 6:

(A) According to Euler’s formula, how many faces would \( K_5 \) need to have in any planar embedding?

(B) Show that \( K_5 \) is not planar. (Hint: Use part (A) and the fact that every face must contain at least three edges. Each edge is adjacent to exactly two faces.)

(C) Repeat parts (A) and (B) for \( K_{3,3} \).

Problem 7:

Find a subdivision of \( K_{3,3} \) or \( K_5 \) in, or a planar drawing of, each of the following graphs.

![Graphs](image-url)