Unsteady Inviscid Flows

N-S: \( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x} \)

For irrotational flows, \( u = \frac{\partial \phi}{\partial x} \) and \( \nabla \times \mathbf{u} = 0 \), so that
\[
\mathbf{u} \cdot \nabla \left[ \rho \frac{\partial \phi}{\partial t} + \frac{1}{2} \rho |\nabla \phi|^2 + p + \gamma \right] = 0
\]

which yields the **TIME-DEPENDENT BERNOULLI EQUATION**:
\[
\rho \frac{\partial \phi}{\partial t} + \frac{1}{2} \rho |\nabla \phi|^2 + p + \gamma = F(t)
\]

**WATER WAVES**

\[ z = 0 \quad z = -h \]

Motion of fluid may be described to leading order as time-dependent, inviscid, and irrotational.

Normal Vector: \( \mathbf{n} = \left( \frac{-y, 1}{(1+y^2)^{1/2}} \right) \)

Curvature: \( \mathbf{n} \cdot \nabla = \frac{-y_{xx}}{(1+y^2)^{3/2}} \)

Must deduce a solution for the velocity potential \( \phi \) satisfying
\[
\nabla^2 \phi = 0 \quad \text{subject to kinematic and dynamic B.C.s}
\]

- \( \frac{\partial \phi}{\partial z} = 0 \) on \( z = -h \)
- Kinematic: \( \mathbf{u} = \frac{\partial \phi}{\partial x} \mathbf{i} + \frac{\partial \phi}{\partial y} \mathbf{j} \) on \( z = h \)
- \( \frac{1}{(1+y^2)^{1/2}} \frac{\partial y}{\partial x} = -\frac{\partial x}{\partial y} \frac{\partial \phi}{\partial x} + \frac{1}{(1+y^2)^{1/2}} \frac{\partial \phi}{\partial y} \frac{\partial y}{\partial t} = \mathbf{u}_2 \)
\[ \frac{\partial Y}{\partial t} = U_x \Rightarrow \frac{\partial Y}{\partial t} + U_x \frac{\partial Y}{\partial x} = U_x \]

\[ \Rightarrow \frac{\partial Y}{\partial t} + \frac{\partial \phi}{\partial x} \frac{\partial Y}{\partial x} = \frac{\partial \phi}{\partial x} \quad \text{on} \quad z = y \]

3. **Dynamic B.C.** (Time-dep Bernoulli applied at free surface)

\[ \rho \frac{\partial \phi}{\partial t} + \frac{1}{2} \rho |\nabla \phi|^2 + \rho g Y + P_s = f(t) \quad \text{indep of } x \]

where \( P_s = P_0 + \sigma \frac{y}{h} \) is the surface pressure.

\[ = P_0 - \sigma \frac{y}{(1 + y^2)^{3/2}} \]

Now consider small-amplitude waves and linearize the system of equations and boundary conditions (i.e. assume \( \phi, Y \) small, so neglect any terms involving \( \phi^2, Y^2 \) or \( \phi Y \) or their derivatives)

\[ \nabla^2 \phi = 0 \quad \text{in } -h \leq z \leq 0 \]

b.c.s 1. \( \frac{\partial \phi}{\partial z} = 0 \quad \text{on } z = -h \)

2. \( \frac{\partial Y}{\partial z} = \frac{\partial \phi}{\partial z} \quad \text{on } z = 0 \)

3. \( \rho \frac{\partial \phi}{\partial t} + \rho g Y + P_0 - \sigma y \frac{\partial Y}{\partial x} = f(t) \quad \text{on } z = 0 \)

Seek solns of the form

\[ Y = Y e^{ik(x-ct)} \]

\[ \phi = \phi(\xi)e^{ik(x-ct)} \]

Travelling waves in \( x \)-direction with phase speed \( C \) and wavenumber \( k \)

Sub in \( \phi \) into harmonic eqn:

\[ \nabla^2 \phi = -k^2 \phi = 0 \]

\( \Rightarrow \) Solns are \( \phi(z) = e^{kz}, e^{-kz} \) or \( \sinh kz, \cosh kz \)

We may satisfy B.C. 1 \( \frac{\partial \phi}{\partial z} = 0 \) on \( z = -h \) by choosing

\[ \phi(\xi) = A \cosh k(z+h) \]

\( A \) a constant
New B. 2 \Rightarrow -ikc \hat{\mathbf{J}} = AK \sinh kh \quad \# \\
B. 3 \Rightarrow (-ikc\rho A \cosh kh + \rho g \hat{\mathbf{J}} + k^2 \sigma \hat{\mathbf{J}})^{\text{i}k(x-c)} = f(t) \quad \text{indep of } x \\
\text{i.e. } -ikc\rho A \cosh kh + \rho g \hat{\mathbf{J}} + k^2 \sigma \hat{\mathbf{J}} = 0 \\
\Rightarrow A = -i\frac{\rho g \hat{\mathbf{J}}}{\sinh kh} \Rightarrow \text{sub into } \#: \\
\bbox[2pt,2pt,black]{c^2 = \left(\frac{g}{k} + \frac{\sigma k^3}{\rho^2}\right) \tanh kh} \\
\text{PHASE SPEED}

Since \( c = \frac{w}{k} \) is the relation between phase speed \( c \), wave number \( k = \frac{2\pi}{\lambda} \) and frequency \( \omega \), we thus have
\bbox[2pt,2pt,black]{\omega^2 = \left(gk + \frac{\sigma k^3}{\rho^2}\right) \tanh kh} \\
\text{DISPERSION RELATION}

\text{NB: as } k \to \infty, \tanh k \to 1

\text{PHysical Interpretation:}

- the relative importance of surface tension and gravity is prescribed by the Bond number \( B_0 \)
  \[
  B_0 = \frac{\sigma k^2}{\rho g} = \frac{\rho (2\pi)^2}{\rho g} = \frac{\lambda^2}{\lambda^2}
  \]
- for air-water, \( B_0 \sim 1 \) for \( \lambda \sim 1.7 \text{ cm} \) \text{(CAPILLARY)}
- for \( \lambda \gg \lambda_c \), surface tension effects negligible \( (B_0 \gg 1) \) \text{ "gravity waves"}
- for \( \lambda \ll \lambda_c \), influence of gravity is negligible \( (B_0 \ll 1) \) \text{ capillary waves}

\[
\text{Note: } c \text{ predicts } \lambda = \frac{2\pi}{k} \\
C_{\text{min}} = \left(\frac{4g\sigma}{\rho}\right)^{\frac{1}{2}} \\
\text{for } k = \left(\frac{\rho g}{\sigma}\right)^{\frac{1}{2}}
\]
SPECIAL CASES

B_0 > 1

A. Gravity waves:

\[ C^2 = \frac{g}{k} \tanh kh \]

a.) Shallow water (kh < 1) \[ \Rightarrow C = \sqrt{gh} \]

- all wavelengths travel at same speed (i.e. NON-DISPERSIVE)
- one can only surf in shallow water

b.) Deep water (kh >> 1) \[ \Rightarrow C = \sqrt{\frac{g}{k}} \]

- long waves travel fastest
- e.g. drop stone in pond

B. Capillary waves:

B_0 << 1, \[ C^2 = \frac{g}{k} \tanh kh \]

a.) Deep water: \[ k \ll k \Rightarrow C = \sqrt{\frac{gk}{\rho}} \]

- short waves travel fastest
- e.g. raindrop in pond

b.) Shallow waves: \[ k \gg k \Rightarrow C = \sqrt{\frac{gkh^2}{\rho}} \]

Note: in laboratory modelling of shallow water waves (kh < 1)

\[ C^2 = \left( \frac{1}{k} + \frac{g}{k} \right) \left( k^2h - \frac{1}{3} k^3h^2 + O(kh)^5 \right) = gh + \left( \frac{gk}{\rho} - \frac{1}{3} gh^2 \right) k^2 + O(kh)^5 \]

In ripple tanks, choose \[ k = \left( \frac{3g}{h^2} \right)^{\frac{1}{2}} \] to get good approximation to non-dispersive waves. In water \[ \left( \frac{3g}{h^2} \right)^{\frac{1}{2}} \approx \left( \frac{3 \times 9.8}{10^3} \right)^{\frac{1}{2}} \approx 5 \text{ cm} \]
**Summary**

\[ C_{\text{min}} = \left( \frac{4 \cdot \rho}{\rho} \right) \]

\[ \text{Note: } C_{\text{min}} = \left( \frac{4 \cdot \rho}{\rho} \right) \]

\[ \text{for } k = \left( \frac{2 \omega}{C} \right) \]

**GROUP VELOCITY**

When \( C = C(h) \), a wave is called dispersive since the different (Fourier) wave components (corresponding to different \( k \) and \( 
\omega \)) separate or disperse.

*Deep Water Gravity Wave*: \( C = \sqrt{g \lambda} \)

- In a dispersive system, the energy of a wave component does not propagate at \( C = \omega/k \) but at the group velocity \( C_g = \frac{dk}{d\omega} \).

*Deep Water Gravity Wave*: \( C_g = \frac{d\omega}{dk} = \frac{g}{2\zeta} \sqrt{\frac{g}{\zeta}} \)

*Deep Water Capillary Wave*: \( C_g = \frac{g^2}{2k^2} \left( C = \sqrt{\frac{g}{2k}} \right) \)

Note: \( C_{\text{min}} \)

If \( U < C_{\text{min}} \): no steady waves generated by obstacle.

If \( U > C_{\text{min}} \): there are 2 \( k \)-values for which \( C = U \).

i) The smaller represents a gravity wave with \( C_g = \frac{C}{2} < C \),

\( \Rightarrow \) energy swept downstream.

ii) The larger represents a capillary wave with \( C_g = \frac{C}{2} > C \),

\( \Rightarrow \) energy transported upstream (but quickly dissipated due to small \( \zeta \)).

**Fishing Line**