18.315 Problem Set 5 (due Tuesday, December 12, 2006.)

Turn in at most 5 problems.

1. A domino tableau is a ribbon tableau of weight \((2, 2, \ldots, 2)\). Let \(DT(\lambda)\) be the number of domino tableaux of shape \(\lambda\). Find a closed expression for the sum \(\sum_\lambda DT(\lambda)^2\) over partitions \(\lambda\) such that \(|\lambda| = 2n\).

2. For partitions \(\lambda, \mu, \gamma\) with \(k\) parts, prove that the Kostka number equals the Littlewood-Richardson coefficient \(K_{\lambda\mu} = c_{\lambda\gamma}^{\mu}\), if \(\gamma\) satisfies the condition \(\min |\gamma_i - \gamma_{i+1}| > \mu_1\). (For example, the equality \(K_{\lambda\mu} = c_{\lambda\gamma}^{\mu}\) holds if \(\gamma = (kN, (k-1)N, \ldots, N)\) for sufficiently large \(N\).)

3. Construct a bijection between two variants \(BZ_1\) and \(BZ_2\) of Berenstein-Zelevinsky triangles. (\(BZ_1\) involves the hexagon condition and \(BZ_2\) has the tail-sum condition.)

4. Construct a bijection between the set of Littlewood-Richardson tableaux \(LR(\lambda/\mu, \nu)\) and the set of Knutson-Tao honeycombs with boundary rays given by \(\lambda, \mu,\) and \(\nu\) (as described in class).

5. Prove Knutson-Tao’s puzzle version of the LR-rule. (It is enough to show that the puzzle LR-rule is equivalent to another version: LR-tableaux, BZ-triangles, or KT-honeycombs.)

6. Let \((\lambda/\mu)\) be the skew shape \(\lambda/\mu\) rotated by \(180^\circ\). Construct a bijection between the sets of Littlewood-Richardson tableaux \(LR(\lambda/\mu, \nu)\) and \(LR((\lambda/\mu)\), \(\nu)\).

7. Let \(V_\lambda\) be the irreducible representations of \(S_n\) labelled by partitions \(\lambda\) as in the Okounkov-Vershik construction. (That is the eigenvalues of the Jucys-Murphy elements in the representation \(V_\lambda\) are the contents of the shape \(\lambda\).) Also let \(\tilde{V}_\lambda\) be the irreducible representation of \(S_n\) whose character \(\chi_\lambda\) corresponds to the Schur function \(s_\lambda\) under the Frobenius characteristic map \(ch\). Prove that \(V_\lambda = \tilde{V}_\lambda\). In other words, show that Okounkov-Vershik’s and Frobenius’ approaches lead to the same labelling of the irreducible representations by partitions.

8. (a) Prove that \(\sum_\lambda z_\lambda^{-1} p_\lambda(x)p_\lambda(y) = \prod_{i,j} \frac{1}{1-x_iy_j}\) and deduce that \(\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda\mu}\). (b) Prove that \(\sum_{\lambda:|\lambda|=n} z_\lambda^{-1} p_\lambda = h_n\).

9. Calculate all values of the character \(\chi_{(n-1,1)}\) of the irreducible representation \(V_{(n-1,1)}\) of \(S_n\).

10. For a partition \(\lambda\), give a closed formula for the character value \(\chi_\lambda((2,1,\ldots,1))\).