1. Let \( f^\lambda \) be the number of standard Young tableaux of shape \( \lambda \).

(a) (5 points) Prove that the sum \( \sum (f^\lambda)^2 \) over all partitions \( \lambda \) of \( n \) with at most 2 parts (that is \( \lambda = (\lambda_1, \lambda_2), \lambda_1 \geq \lambda_2 \geq 0, \lambda_1 + \lambda_2 = n \)) equals the Catalan number \( C_n = \frac{1}{n+1} \binom{2n}{n} \). (You can use the combinatorial interpretation of \( C_n \) as the number of Dyck paths.)

(b) (5 points) Find (and prove) a closed formula for the sum \( \sum f^\lambda \) over partitions \( \lambda \) of \( n \) with at most 2 parts. The formula might involve a summation.

2. Let \( V := \{(z_1, \ldots, z_n) \mid z_1 + \cdots + z_n = 0\} \cong \mathbb{C}^{n-1} \). The symmetric group \( S_n \) acts on \( V \) by permutations of the coordinates.

(a) (5 points) Find the Gelfand-Tsetlin basis of the representation \( V \).

Hint: Find the basis \( v_1, \ldots, v_{n-1} \) of \( V \) such that each \( v_i \) is a common eigenvector of the Jucys-Murphy elements \( X_i = (1, i) + (2, i) + \cdots + (i-1, i) \in \mathbb{C}[S_n] \), for \( i = 1, \ldots, n-1 \).

(b) (5 points) Prove that \( V \) is equivalent to a certain irreducible representation \( V_\lambda \) of \( S_n \) and identify the partition \( \lambda \).

Hint: Look at eigenvalues of the Jucys-Murphy elements and use the correspondence with content vectors of Young tableaux.

3. (a) (5 points) Prove that the Jucys-Murphy elements \( X_i \) and \( X_j \) commute with each other (that it \( X_i X_j = X_j X_i \)) using only the definition of these elements.

(b) (5 points) Let \( \text{Cyc}_n \) be the element the group algebra \( \mathbb{C}[S_n] \) given by \( \text{Cyc}_n = \sum w \) over all permutations \( w \in S_n \) with a single cycle of size \( n \). Express \( \text{Cyc}_n \) in terms of the Jucys-Murphy elements for \( n = 1, 2, 3, 4 \).

(c) (5 points) Express \( \text{Cyc}_n \) in terms of the Jucys-Murphy elements for an arbitrary \( n \).

(d) (5 points) It is clear that \( X_1 = 0, X_2^2 = 1 \). Check that \( X_3^3 = 3X_3 + 2X_2 \). For any \( i \), express some power \((X_i)^d \) as a polynomial in \( f(X_1, \ldots, X_i) \) of degree \( \deg f < d \).

4. (10 points) Let \( T \) be a rooted tree on \( n \) nodes. Prove the following “baby hooklength formula:”

\[
ext(T) = \frac{n!}{\prod_{v \in T} h(v)}.
\]

Here \( \text{ext}(T) \) is the number of linear extensions of \( T \), that is \( \text{ext}(T) \) is the number of ways to label the nodes of \( T \) by \( 1, \ldots, n \) so that, for each node labeled \( i \), all children of this node have labels greater than \( i \). The
“hooklength” \( h(v) \) of a node \( v \) in \( T \) is the total number of descendants of \( v \) (including the node \( v \) itself).

5. (a) (5 points) An \textit{involution} is a permutation \( w \in S_n \) such that \( w^2 = 1 \) (that is \( w \) has only cycles of sizes 1 or 2). Prove that the number of involutions in \( S_n \) equals

\[
I_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{2^k k!(n - 2k)!}.
\]

(b) (5 points) We know that \( \sum_{|\lambda|=n} (f^\lambda)^2 = n! \). Prove that the sum \( \sum_{|\lambda|=n} f^\lambda \) equals the number \( I_n \) of involutions \( w \in S_n \).

6. A \textit{skew Young diagram} \( \kappa = \lambda/\mu \) is the set-theoretic difference of two usual Young diagrams shapes \( \lambda \) and \( \mu \). For example, \( \lambda/1 \) is the Young diagram of shape \( \lambda \) with the top left box removed. One can define standard Young tableaux for skew shapes in the usual way as fillings of boxes with numbers \( 1, \ldots, n \) that increase in rows and columns. Let \( f^\kappa \) be the number of such skew Young tableaux.

A \textit{ribbon} is a skew Young diagram such that that (i) it has a single connected component, and (ii) it contains no \( 2 \times 2 \)-box inside. (We consider ribbons up to parallel translations.) For example, there are

2 ribbons with 2 boxes: \( \begin{array}{|c|c|} \hline \end{array}, \begin{array}{|c|c|} \hline \end{array} \); 4 ribbons with 3 boxes: \( \begin{array}{|c|c|c|} \hline \end{array}, \begin{array}{|c|c|c|} \hline \end{array}, \begin{array}{|c|c|} \hline \end{array}, \begin{array}{|c|c|} \hline \end{array} \); etc.

(a) (5 points) Find the number of ribbons with \( n \) boxes.

(b) (5 points) Find the sum \( \sum f^\kappa \), where \( \kappa \) varies over all ribbons with \( n \) boxes.

(c)\( ^* \) (10 points) For given \( n \), find a ribbon \( \kappa \) with \( n \) boxes such that \( f^\kappa \) has the maximal possible value (among all ribbons with \( n \) boxes). Prove that this is the maximal possible value.

7. A \textit{horizontal \( k \)-strip} is a skew Young shape with \( k \) boxes that contains no two boxes in the same column. (It may contain several connected components.)

Let \( U_k \) and \( D_l \) be the operators that act on the space \( \mathbb{C}^Y \) of linear combinations of Young diagrams, as follows. \( U_k : \lambda \mapsto \sum \mu \), there the sum is over all \( \mu \) obtained from \( \lambda \) by adding a horizontal \( k \)-strip. \( D_l : \lambda \mapsto \sum \mu \) there the sum is over all \( \mu \) obtained from \( \lambda \) by removing a horizontal \( l \)-strip. In particular, \( U_1 \) and \( D_1 \) are the “up” and “down” operators for the Young lattice.

(a) (10 points) Prove that, for any \( k, l \geq 0 \),

\[
U_k U_l = U_l U_k, \quad D_k D_l = D_l D_k.
\]
\[ D_k U_l = \sum_{r=0}^{\min(k,l)} U_{l-r} D_{k-r} \]

(b)* (10 points) Use these operations to give an alternative proof of the fact that the number of pairs \((P, Q)\) of semi-standard Young tableaux of the same shape and with weights \((\beta_1, \beta_2, \ldots)\) and \((\gamma_1, \gamma_2, \ldots)\) equals the number of matrices \(A = (a_{ij})\) with nonnegative integer entries, with row sums \(\sum_j a_{ij} = \beta_i\) and column sums \(\sum_i a_{ij} = \gamma_j\) (as in RSK-correspondence).

8. Fix two sequences of integers \(r_1, \ldots, r_n\) and \(c_1, \ldots, c_n\). Let \(S_1\) be the set of nonegative integer \(n \times n\)-matrices \(A = (a_{ij})\) with given row sums \(\sum_j a_{ij} = c_i\) and column sums \(\sum_i a_{ij} = r_j\). Let \(S_2\) be the set of nonnegative integer \(n \times n\)-matrices \(B = (b_{ij})\) such the entries weakly decrease in the rows and in the columns (that is \(b_{ij} \geq b_{i',j'}\) whenever \(i \leq i'\) and \(j \leq j'\)) and the diagonal sums \(d_k = \sum_{j-i=k} b_{ij}\) are equal to \(d_{n-i} = r_1 + \cdots + r_i\) and \(d_{-n+i} = c_1 + \cdots + c_i\), for \(i = 1, \ldots, n\).

(a) (5 points) Construct an explicit bijection between \(S_1\) and \(S_2\) for \(n = 2, 3\).

(b) (5 points) Prove that \(|S_1| = |S_2|\), for any \(n\).

9*. (10 points) In class we constructed the tranformations of semi-standard Young tableaux \(s_i : T \mapsto \tilde{T}\) such that (1) \(T\) and \(\tilde{T}\) have the same shape, and (2) if the weight of \(T\) is \((\beta_1, \ldots, \beta_i, \beta_{i+1}, \ldots)\) then the weight of \(\tilde{T}\) is \((\beta_1, \ldots, \beta_{i+1}, \beta_{i}, \ldots)\). Modify these operations and define new operations \(s_i\) acting on semi-standard tableaux that satisfy the above properties and, in addition, satisfy the Coxeter relations:
\[ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad s_i^2 = 1, \quad s_i s_j = s_j s_i \text{ for } j \neq i \pm 1. \]

Then these operations can be extended to the action of the symmetric group on semi-standard tableaux by setting \(w(T) := s_{i_1} \cdots s_{i_l}(T)\) for a permutation \(w = s_{i_1} \cdots s_{i_l}\).