NONLINEAR SYSTEMS

1 Systems of first order ODEs

Consider the autonomous system of ODE’s

\[
\begin{align*}
\frac{dx_1}{dt} &= f_1(x_1, \ldots, x_n) \\
\frac{dx_2}{dt} &= f_2(x_1, \ldots, x_n) \\
&\vdots \\
\frac{dx_n}{dt} &= f_n(x_1, \ldots, x_n)
\end{align*}
\]

1. The critical points of the systems are the solutions \((x_1, \ldots, x_n)\) of the system

\[
\begin{align*}
f_1(x_1, \ldots, x_n) &= 0 \\
f_2(x_1, \ldots, x_n) &= 0 \\
&\vdots \\
f_n(x_1, \ldots, x_n) &= 0
\end{align*}
\]

2. The Jacobian of the system is given by the \(n \times n\) matrix

\[
J(x_1, \ldots, x_n) = \begin{bmatrix}
\frac{df_1}{dx_1} & \frac{df_1}{dx_2} & \cdots & \frac{df_1}{dx_n} \\
\frac{df_2}{dx_1} & \frac{df_2}{dx_2} & \cdots & \frac{df_2}{dx_n} \\
&\vdots & \ddots & \vdots \\
\frac{df_n}{dx_1} & \frac{df_n}{dx_2} & \cdots & \frac{df_n}{dx_n}
\end{bmatrix}.
\]

3. The behavior around a critical point \((a_1, \ldots, a_n)\) is given by the behavior of linearization at that critical point. Namely, by the linear system

\[
u' = J(a_1, \ldots, a_n)u.
\]
How to study the behavior of a nonlinear system

1. Find its critical points.

2. Find its Jacobian.

3. For each critical point \((a_1, \ldots, a_n)\):
   
   (a) Compute the Jacobian at that point, i.e. the matrix \(A = J(a_1, \ldots, a_n)\).
   
   (b) Find the eigenvalues and eigenvectors of \(A\).

   (c) Determine the nature of the critical point and the stability of the system around it. If your system is \(2 \times 2\), determine the nature of the critical point (sink, source, saddle, etc...) and draw the phase portrait around it. (Only if asked!)

4. Again for a \(2 \times 2\) system, one can determine the long-term behavior of the solutions by trapping them in a box containing all the critical points.