WHEN DO RANDOM WALKS RETURN?

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§0: BASICS OF PROBABILITY THEORY

Ever since Kolmogorov’s renowned work in the 1920’s, the standard model for probability consists of three components: a sample space \( \Omega \), a collection \( \mathcal{F} \) of subsets \( A \subseteq \Omega \) called events, and a probability measure \( \mathbb{P} \) which assigns to each \( A \in \mathcal{F} \) its probability \( \mathbb{P}(A) \in [0,1] \). It is always assumed that

1. \( \Omega \) is not empty,
2. \( \Omega \in \mathcal{F} \), \( A \in \mathcal{F} \implies A^c = \Omega \setminus A \in \mathcal{F} \), and \( \{A_n\}_1^\infty \subseteq \mathcal{F} \implies \bigcup_{n=1}^\infty A_n \in \mathcal{F} \).
3. \( \mathbb{P}(\Omega) = 1 \) and if \( \{A_n\}_1^\infty \subseteq \mathcal{F} \) with \( A_n \cap A_{n'} = \emptyset \) for \( n \neq n' \), then

\[
\mathbb{P}\left(\bigcup_{n=1}^\infty A_n\right) = \sum_{n=1}^\infty \mathbb{P}(A_n).
\]

A random variable is a real-valued function \( X \) on \( \Omega \) with the property that, for each \( \lambda \in \mathbb{R} \), \( \{\omega : X(\omega) \leq \lambda\} \in \mathcal{F} \). When \( X \) takes all its values from a countable set \( \{a_k\}_1^\infty \) and \( f : \mathbb{R} \rightarrow \mathbb{R} \), we call

\[
\mathbb{E}[f(X)] = \sum_{k=1}^\infty f(a_k) \mathbb{P}(X = a_k)
\]

the expected value of \( f(X) \). In general, one has to worry about the convergence of such sums. However, if \( f \) is non-negative, the only thing which could go wrong is that the sum might be \(+\infty\), which is disturbing but unambiguous. It is important to know that if \( X \) and \( Y \) are random variables and \( f \) and \( g \) are functions, then \( \mathbb{E}[f(X) + g(Y)] = \mathbb{E}[f(X)] + \mathbb{E}[g(Y)] \). In addition, we will make frequent applications of Markov’s Inequality:

\[
\mathbb{P}\{|f(X)| \geq \lambda\} \leq \frac{1}{\lambda} \mathbb{E}[f(X)] \quad \text{for all } \lambda \in (0, \infty),
\]

which comes from the computation

\[
\mathbb{E}[|f(X)|] \geq \sum_{\{\ell : |f(a_\ell)| \geq \lambda\}} |f(a_\ell)| \mathbb{P}(X = a_\ell) \geq \lambda \sum_{\{\ell : |f(a_\ell)| \geq \lambda\}} \mathbb{P}(X = a_\ell) = \lambda \mathbb{P}(\{|f(X)| \geq \lambda\}).
\]

Finally, two events \( A \) and \( B \) will be said to be independent if \( \mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B) \).

Warning: Throughout what follows, we will not worry about which subsets of \( \Omega \) are events. In other words, we will proceed as if all subsets are elements of \( \mathcal{F} \). This is a questionable assumption, but it turns out to cause no harm in the considerations below.

\[1\] Here, and elsewhere, we use the shorthand \( \mathbb{P}(X = a) \) to denote \( \mathbb{P}(\{\omega : X(\omega) = a\}) \).
§1: Random Walks on $\mathbb{Z}$

Here we will model a random walk in the integers $\mathbb{Z}$ by taking our sample space $\Omega$ to consist of all strings $\omega = (\omega_1, \ldots, \omega_n, \ldots)$, where, for each $n \geq 1$, $\omega_n \in \{-1, 1\}$. In this model, one can think of $\omega$ as recording the steps taken during a random walk: $\omega_n = 1$ if the walker went forward on the $n$th step and $\omega_n = -1$ if he went backward. Further, we will assume that what he does on any given step is independent of what he does on any of the other steps but has the same distribution as it does on any other step. Equivalently, we are saying that there is a $p \in [0, 1]$ such that, for any $n \geq 1$ and $(\epsilon_1, \ldots, \epsilon_n) \in \{-1, 1\}^n$,

$$P\{\omega : \omega_1 = \epsilon_1, \ldots, \omega_n = \epsilon_n\} = p^{N^+}(\epsilon)q^{N^-}(\epsilon),$$

where $q \equiv 1 - p$ and

$$N^\pm(\epsilon) = \sum_{1 \leq m \leq \lfloor \epsilon \rfloor} 1(\pm 1)^m$$

is the number of $1 \leq m \leq n$ for which $\epsilon_m = \pm 1$. In order to eliminate trivial, but annoying, cases, we will assume throughout that $p /\in \{0, 1\}$ (i.e., $0 < p < 1$).

Next, let $S_0(\omega) = 0$ and $S_n(\omega) = \sum_{m=1}^{\lfloor \omega \rfloor} \omega_m$ for $n \geq 1$

be the position of the random walker at time $n \geq 0$. Then:

$$P(S_n = k) = \left(\frac{n}{n+k}\right)^{n+k} q^{n-k}$$

if $n + k$ is an even integer between 0 and $2n$ and is 0 otherwise. In addition,

$$E[S_n] = \sum_{m=1}^{\lfloor \omega \rfloor} E[\omega_m] = n\mu \quad \text{where } \mu = p - q = E[1].$$

Lemma. For any $p \in (0, 1)$,

\begin{equation}
E[(S_n - n\mu)^2] = 4npq \quad \text{for all } n \geq 1.
\end{equation}

In addition,

\begin{equation}
E[(S_n - n\mu)^4] \leq 6n^2 \quad \text{for all } n \geq 1.
\end{equation}

Proof. Observe that $S_n - n\mu = \sum_{m=1}^{\lfloor \omega \rfloor} (\omega_m - \mu)$ and therefore that

$$E[(S_n - n\mu)^2] = \sum_{m=1}^{\lfloor \omega \rfloor} E[(\omega_m - \mu)^2] + \sum_{m_1, m_2=1 \atop m_1 \neq m_2} E[(\omega_{m_1} - \mu)(\omega_{m_2} - \mu)] = 4npq$$

because

$$E[(\omega_m - \mu)^2] = p4q^2 + q4p^2 = 4pq(p + q) = 4pq$$

and

$$E[(\omega_{m_1} - \mu)(\omega_{m_2} - \mu)] = E[(\omega_{m_1} - \mu)]E[(\omega_{m_2} - \mu)] = 0 \quad \text{for } m_1 \neq m_2.$$
Similarly,
\[
E[(S_n - n\mu)^4] = \sum_{m_1, \ldots, m_4=1}^{m} E[(\omega_{m_1} - \mu) \cdots (\omega_{m_4} - \mu)] \\
= \sum_{m=1}^{n} E[(\omega_m - \mu)^4] + \frac{4}{2} \sum_{m_1, m_2=1}^{m} E[(\omega_{m_1} - \mu)^2(\omega_{m_2} - \mu)^2] \\
= n[p(2q)^4 + q(2p)^4] + 96n(n - 1)p^2q^2 \leq 6n^2. \quad \Box
\]

As a consequence of (0.1) and (1.2), we see that
\[
P(\forall n \geq m \mid S_n - n\mu \geq \frac{1}{n^\frac{3}{4}}) = \mathbb{P}(\forall n \geq m \mid S_n - n\mu \geq n^\frac{3}{8}) \\
\leq \lim_{m \to \infty} \sum_{n=m}^{\infty} \mathbb{P}(S_n - n\mu \geq n^\frac{3}{8}) \leq \lim_{m \to \infty} \sum_{n=m}^{\infty} \frac{1}{n^\frac{3}{8}} \mathbb{E}[(S_n - n\mu)^4] \leq \lim_{m \to \infty} 6 \sum_{m=m}^{\infty} \frac{1}{n^\frac{3}{8}} = 0.
\]

In words, this means that, almost surely\(^3\) there is an \(m \geq 1\) such that \(\frac{S_n}{n} - \mu \leq n^{-\frac{3}{8}}\) for all \(n \geq m\), and therefore
\[
(1.3) \quad \lim_{n \to \infty} \frac{S_n}{n} = \mu \quad \text{almost surely.}
\]

In particular,
\[
(1.4) \quad \lim_{n \to \infty} S_n = \infty \quad (-\infty) \quad \text{if } p > \frac{1}{2} \quad (p < \frac{1}{2}).
\]

**Recurrence and Occupation:** Set \(R_\ell\) denote the set of \(\omega\) for which there are at least \(\ell\ \ n \geq 1\) with \(S_n(\omega) = 0\). That is, \(R_\ell\) is the event that the walker returns to 0 at least \(\ell\) times.

**Theorem.** For each \(\ell \geq 1\), \(\mathbb{P}(R_\ell) = \mathbb{P}(R_1)^\ell\).

**Proof.** Let \(\rho_\ell(\omega)\) be the time of the \(\ell\)th return to 0 \((\rho_\ell(\omega) \equiv \infty\) if \(\omega \notin R_\ell\)). Then
\[
\mathbb{P}(R_{\ell+1}) = \mathbb{P}(\{\rho_\ell < \infty\} \cap R_{\ell+1}) = \sum_{m=1}^{\infty} \mathbb{P}(\{\rho_\ell = m\} \cap \{\exists n > m \ S_n = 0\}) \\
= \sum_{m=1}^{\infty} \mathbb{P}(\{\rho_\ell = m\} \cap \{\exists n > m \ S_n - S_m = 0\}) = \sum_{m=1}^{\infty} \mathbb{P}(\{\rho_\ell = m\}) \mathbb{P}(R_1) = \mathbb{P}(R_\ell) \mathbb{P}(R_1). \quad \Box
\]

Let \(T(\omega) \equiv \sum_{n=0}^{\infty} 1_{\{0\}}(S_n(\omega))\) denote the total number of visits that the walker with history \(\omega\) makes to 0.

**Corollary.** Either \(\mathbb{E}[T] < \infty\), in which case \(T < \infty\) almost surely, or \(\mathbb{E}[T] = \infty\), in which case \(T = \infty\) almost surely. Moreover, \(\mathbb{E}[T] = \infty \iff \mathbb{P}(R_1) = 1\). In fact,
\[
(1.5) \quad \mathbb{E}[T] = \frac{1}{1 - \mathbb{P}(R_1)}.
\]

\(^3\)The term “almost surely” is applied to an event which occurs with probability 1.
Proof. Suppose that $\mathbb{P}(R_1) = 1$. Then $\mathbb{P}(R_\ell) = 1$ and therefore $\mathbb{P}(T \geq \ell) = 1$ for all $\ell \geq 1$. Hence,

$$\mathbb{P}(R_1) = 1 \implies T \geq \ell \text{ for all } \ell \geq 1 \text{ almost surely } \implies T = \infty \text{ almost surely}.$$ 

Next, suppose that $\gamma \equiv \mathbb{P}(R_1) < 1$. Then

$$\mathbb{P}(T \geq \ell + 1) = \mathbb{P}(R_\ell) = \gamma\ell,$$

and so

$$\mathbb{P}(T = \ell) = (1 - \gamma)\gamma^{\ell - 1},$$

from which

$$\mathbb{E}[T] = (1 - \gamma) \sum_{\ell=1}^{\infty} \ell\gamma^{\ell - 1} = (1 - \gamma) \frac{d}{d\gamma} \sum_{\ell=0}^{\infty} \gamma^\ell = (1 - \gamma) \frac{d}{d\gamma} (1 - \gamma)^{-1} = \frac{1}{1 - \gamma}$$

follows immediately. \qed

Corollary. Depending on whether $p > \frac{1}{2}$ or $p < \frac{1}{2}$,

$$\mathbb{P}(\exists n \geq 1 S_n < 0) = \mathbb{P}(\exists n \geq 1 S_n = 0) < 1$$

or

$$\mathbb{P}(\exists n \geq 1 S_n \geq 0) = \mathbb{P}(\exists n \geq 1 S_n = 0) < 1$$

Proof. It is clearly enough to treat the case when $p > \frac{1}{2}$. But, because

$$\lim_{n \to \infty} S_n(\omega) = +\infty \implies T(\omega) < \infty$$

and

$$S_0(\omega) = 0 \text{ and } \lim_{n \to \infty} S_n(\omega) = +\infty \implies \exists n \geq 1 S_n(\omega) \leq 0 \text{ if and only if } \exists n \geq 1 S_n(\omega) = 0,$$

the conclusion follows from the preceding corollary combined with (1.4). \qed

The preceding corollary gives an account of what happens when $p \neq \frac{1}{2}$. In order to see what happens when $p = \frac{1}{2}$, we note that

$$\mathbb{E}[T] = \mathbb{E} \left[ \sum_{n=0}^{\infty} \mathbf{1}_{(0)}(S_n) \right] = \sum_{n=0}^{\infty} \mathbb{P}(S_n = 0) = 1 + \sum_{n=1}^{\infty} \mathbb{P}(S_{2n} = 0),$$

and, therefore by (1.5), that

$$\mathbb{P}(\exists n \geq 1 S_n = 0) = 1 \iff \sum_{n=0}^{\infty} \mathbb{P}(S_{2n} = 0) = \infty.$$
Lemma. When \( p = \frac{1}{2} \), for any \( \ell \in \mathbb{Z} \): \( \mathbb{P}(S_{2n} = \ell) \leq \mathbb{P}(S_n = 0) \). In particular, \( \mathbb{P}(S_{2n} = 0) \geq (4n + 1)^{-1} \).

Proof. Note that \( S_n \) and \( -S_n \) have the same distribution, and apply Schwarz’s inequality to conclude that

\[
\mathbb{P}(S_{2n} = \ell) = \sum_{k \in \mathbb{Z}} \mathbb{P}(S_n = k \& S_{2n} - S_n = \ell - k) = \sum_{k \in \mathbb{Z}} \mathbb{P}(S_n = k) \mathbb{P}(S_n = \ell - k) \\
\leq \left( \sum_{k \in \mathbb{Z}} \mathbb{P}(S_n = k)^2 \right)^{\frac{1}{2}} \left( \sum_{k \in \mathbb{Z}} \mathbb{P}(S_n = \ell - k)^2 \right)^{\frac{1}{2}} \\
= \sum_{k \in \mathbb{Z}} \mathbb{P}(S_n = k)^2 = \sum_{k \in \mathbb{Z}} \mathbb{P}(S_n = k) \mathbb{P}(S_n = -k) = \mathbb{P}(S_{2n} = 0).
\]

Finally, since

\[
1 = \sum_{\ell = -2n}^{2n} \mathbb{P}(S_{2n} = \ell) \leq (4n + 1) \mathbb{P}(S_{2n} = 0),
\]
the last inequality follows. \( \square \)

Theorem. When \( p = \frac{1}{2} \), \( T = \infty \) almost surely, and so, with probability 1, \( S_n = 0 \) for infinitely many \( n \)’s.

Symmetric Random Walk on \( \mathbb{Z}^2 \)

Now \( \Omega \) consists of strings \( \omega = (\omega_1, \ldots, \omega_n, \ldots) \) where, for each \( n \geq 1 \), \( \omega_n \in \{(-1, 0), (1, 0), (0, -1), (0, 1)\} \) and

\[
\mathbb{P}(\omega_1 = \epsilon_1, \ldots, \omega_n = \epsilon_n) = 4^{-n}
\]
for all \( n \geq 1 \) and \( \epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \{(-1, 0), (1, 0), (0, -1), (0, 1)\}^n \). Next, define \( S_0(\omega) = 0 \equiv (0, 0) \) and \( S_n(\omega) = \sum_{m=1}^{n} \omega_m \) for \( n \geq 1 \), and set

\[
T(\omega) = \sum_0^\infty 1_{\{0\}}(S_n(\omega)).
\]

Just as before,

\[
\sum_0^\infty \mathbb{P}(S_{2n} = 0) = \mathbb{E}[T] = \left( \mathbb{P}(\forall n \geq 1 S_n \neq 0) \right)^{-1}, \tag{2.1}
\]

\[
\mathbb{P}(\exists n \geq 1 S_n = 0) = 1 \iff T = \infty \text{ almost surely}, \tag{2.2}
\]

and

\[
\mathbb{P}(S_{2n} = \ell) \leq \mathbb{P}(S_{2n} = 0), \quad \ell \in \mathbb{Z}^2. \tag{2.3}
\]

In particular, from (2.1) and (2.2), we know that

\[
\mathbb{P}(\exists n \geq 1 S_n = 0) = 1 \iff \mathbb{E}[T] = \infty \iff T = \infty \text{ almost surely}. \tag{2.4}
\]

By (2.1) and (2.4), we would know that \( T = \infty \) almost surely if we knew that \( \sum_{n=0}^\infty \mathbb{P}(S_{2n} = 0) = \infty \). By (2.3) plus the fact that \( |S_{2n}| \leq 4n \), we have that \( \mathbb{P}(S_{2n} = 0) \geq (4n + 1)^{-2} \), from which we cannot draw any conclusion. In order to sharpen this estimate, we will need the following.
Lemma. For any $n \geq 1$, 

$$P(|S_n| \leq \sqrt{2n}) \geq \frac{1}{2}. \tag{2.5}$$

Proof. Let $(S_n)_1$ be the first coordinate of $S_n$. Then $(S_n)_1 = \sum_{m=1}^n (\omega_m)_1$, where $(\omega_m)_1$ is the first coordinate of $\omega_m$. Since the $\omega_m$’s are mutually independent, so are the $(\omega_m)_1$’s. Moreover, $P((\omega_m)_1 = \pm 1) = \frac{1}{2}$ and $P((\omega_m)_1 = 0) = \frac{1}{2}$. Thus, just as in the derivation of (1.1),

$$E[(S_n)_1^2] = nE[(\omega_1)_1^2] = \frac{n}{2}.$$ 

Since the same computation applies equally well to the second coordinate of $S_n$, it follows that $E[|S_n|^2] = n$. In particular, by (0.1),

$$P(|S_n| \geq \sqrt{2n}) \leq \frac{1}{2n}E[|S_n|^2] \leq \frac{1}{2}.$$ 

Hence, $P(S_n \leq \sqrt{2n}) = 1 - P(S_n > \sqrt{2n}) \geq \frac{1}{2}$. \hfill \Box

From (2.3) and (2.5):

$$\frac{1}{2} \leq \sum_{|k| \leq 2\sqrt{n}} P(S_{2n} = k) \leq (4\sqrt{n} + 1)^2 P(S_{2n} = 0) \leq (32n + 2)P(S_{2n} = 0).$$

We can now replace the earlier estimate by $P(S_{2n} = 0) \geq (32n + 2)^{-1}$, which shows that $\sum_{n=0}^\infty P(S_{2n} = 0) = \infty$. Thus, we have proved the following.

Theorem. For a symmetric random walk in $\mathbb{Z}^2$, $T = \infty$ almost surely.

**Symmetric Random Walk in $\mathbb{Z}^3$**

The setting is analogous to that for $\mathbb{Z}^2$, only now, the steps $\omega_n$’s take one of the six values

$$(0, 0, -1), (0, 0, 1), (0, -1, 0), (0, 1, 0), (-1, 0, 0), (1, 0, 0),$$

each with the same probability: $\frac{1}{6}$. Proceeding in precisely the same way as we did before, one can show that

$$\sum_{n=0}^\infty P(S_{2n} = 0) = E[T] = \left(P(\forall n \geq S_n \neq 0)\right)^{-1},$$

$$P(\exists n \geq 1 S_n = 0) = 1 \iff T = \infty \text{ almost surely},$$

and

$$P(S_{2n} = \ell) \leq P(S_{2n} = 0) \text{ for all } \ell \in \mathbb{Z}^3.$$

However, this time, the argument just given for $\mathbb{Z}^2$, shows in this case that $P(S_{2n} = 0) \geq \epsilon n^{-\frac{3}{2}}$ for some $\epsilon > 0$, which is inconclusive but leaves open the possibility that, here, $E[T] < \infty$.

In order to settle the question left open by the above, we need to develop an upper bound for $P(S_{2n} = 0)$, and for this purpose we will need two observations. In the first place, it is reasonably clear that $S_n$ in $\mathbb{Z}^3$ should behave somewhat like $(S_n^{(1)}, S_n^{(2)}, S_n^{(3)})$, where the $S_n^{(i)}$’s are three, mutually independent symmetric random walks on $\mathbb{Z}$. In particular, one can hope that $P(S_{2n} = 0)$ is commensurate with $P(S_{2n} = 0)^3$; and, if this is true, then the problem reduces to that of getting an upper bound for $P(S_{2n} = 0)$, for which purpose we will use the following estimate.
Lemma. $\mathbb{P}(S_{2n} = 0) \leq e^{\frac{3}{4} n^{-\frac{1}{2}}}$.  

Proof. Because $\mathbb{P}(S_{2n} = 2m) = \binom{2n}{m}4^{-n}$ for $0 \leq m \leq n$, it is easy to see that 

$$
\frac{\mathbb{P}(S_{2n} = 2m)}{\mathbb{P}(S_{2n} = 0)} = \prod_{\ell=0}^{m-1} \frac{n-\ell}{n+m-\ell} = \prod_{\ell=0}^{m-1} \left(1 - \frac{m}{n+m-\ell}\right) \geq \left(1 - \frac{m}{n}\right)^m.
$$

Since $-\log(1 + x) = \sum_{k=1}^{\infty} \frac{x^k}{k}$, we have $x + x^2 \sum_{k=0}^{\infty} \frac{x^k}{k+1} \leq \frac{3}{2}$ for $0 \leq x \leq \frac{1}{2}$, it follows that $\mathbb{P}(S_{2n} = 2m) \geq e^{-\frac{3}{4}} \mathbb{P}(S_{2n} = 0)$ for $0 \leq m \leq \sqrt{n}$ and $n \geq 4$. Finally, if $\lceil \sqrt{n} \rceil$ is the integer part of (i.e., the largest integer dominated by) $\sqrt{n}$, then the preceding shows that 

$$
1 \geq \sum_{m=0}^{\lceil \sqrt{n} \rceil} \mathbb{P}(S_{2n} = m) \geq e^{-\frac{3}{4} \sqrt{n}} \mathbb{P}(S_{2n} = 0). \quad \square
$$

In view of the preceding estimate, we will know that $\mathbb{P}(S_{2n} = 0) \leq C n^{-\frac{3}{2}}$ if we can verify our conjecture about the relationship between random walks in one and three dimensions. Thus, let $\{S_{n}^{(i)} : n \in \mathbb{N}\}$, $i \in \{1, 2, 3\}$, be three mutually independent, symmetric random walks in $\mathbb{Z}$. Next, introduce mutually independent random variables $X_m$, $m \geq 1$, which are independent of all the $S_{n}^{(i)}$’s and distributed so that $\mathbb{P}(X_m = i) = \frac{1}{3}$ for all $m \geq 1$ and $i \in \{1, 2, 3\}$. Set 

$$
N_{0}^{(i)} = 0 \quad \text{and} \quad N_{n}^{(i)} = \sum_{m=1}^{n} 1_{(i)}(X_m) \text{ for } n \geq 1.
$$

One can then verify that $S_{n}$ has the same distribution as 

$$
P(\{S_{n}^{(1)}, S_{n}^{(2)}, S_{n}^{(3)}\}).
$$

Hence, 

$$
\mathbb{P}(S_{2n} = 0) \leq \mathbb{P}\left(\max_{1 \leq i \leq 3} |N_{2n}^{(i)} - \frac{2n}{3}| \geq \frac{n}{3}\right) + \left(\max_{\frac{n}{3} \leq m \leq n} \mathbb{P}(S_m = 0)\right)^3,
$$

which, together with the above, leads to the existence of a $C < \infty$ for which 

$$
\mathbb{P}(S_{2n} = 0) \leq 3\mathbb{P}\left(|N_{2n}^{(i)} - \frac{2n}{3}| \geq \frac{n}{3}\right) + C n^{-\frac{3}{2}}.
$$

Finally, because $2N_{n}^{(i)} - n$ has the same distribution as a random walk on $\mathbb{Z}$ with $p = \frac{1}{3}$, (0.1) together with the estimate in (1.2) tells us that 

$$
\mathbb{P}\left(|N_{n}^{(i)} - \frac{n}{3}| \geq \frac{n}{6}\right) \leq \frac{6^5 n^2}{n^4} = \frac{6^5}{n^2}, \quad n \geq 1.
$$

Thus, we have now proved the first part of the following.

Theorem. For a symmetric random walk in $\mathbb{Z}^d$, $\mathbb{E}[T] < \infty$ and so $\mathbb{P}(\exists n \geq 1 \text{ } S_n = 0) < 1$. In fact, 

$$
\lim_{n \to \infty} |S_n| = \infty \text{ almost surely.}
$$

Proof. To prove the second assertion, set $T_{\{k\}} = \sum_{m=0}^{\infty} 1_{\{k\}}(S_n)$ and $\tau_k = \inf\{n \geq 0 : S_n = k\}$. Then 

$$
\mathbb{E}[T_{\{k\}}] = \sum_{m=0}^{\infty} \mathbb{E}\left[\sum_{n=m}^{\infty} 1_{\{k\}}(S_n - S_m) \text{, } \tau_k = m\right] = \sum_{m=0}^{\infty} \mathbb{E}[T] \mathbb{P}(\tau_k = m) = \mathbb{E}[T] \mathbb{P}(\tau_k < \infty) \leq \mathbb{E}[T].
$$

Hence, if $T_L = \sum_{|k| \leq L} T_{\{k\}}$ is the number of $n$’s with $|S_n| \leq L$, then 

$$
\mathbb{E}[T_L] \leq (2L + 1)^3 \mathbb{E}[T] < \infty \quad \text{for all } L > 0.
$$

But this means that, almost surely, 

$$
\lim_{n \to \infty} |S_n| \geq L \text{ for all } L > 0. \quad \square
$$
Problem #1: Referring to the symmetric random walk on \( Z \), define \( \tau_k(\omega) \) for \( k \in Z \) to be the first time \( n \) that \( S_n(\omega) \) gets to \( k \). That is,

\[
\tau_k(\omega) \equiv \inf \{ n \geq 0 : S_n(\omega) = k \} (\equiv \infty \text{ if } S_n(\omega) \neq k \text{ for any } n \geq 0).
\]

In addition, define, as in the proof of the theorem on p. 3, \( \rho_1(\omega) \) be the first time of return to 0. That is,

\[
\rho_1(\omega) \equiv \inf \{ n \geq 1 : S_n(\omega) = 0 \} (\equiv \infty \text{ if } S_n(\omega) \neq 0 \text{ for any } n \geq 1).
\]

(a) By thinking about what happens on the first step of a random walk, show that

\[
\mathbb{P}(\rho_1 = n) = \frac{1}{2} \mathbb{P}(\tau_1 = n - 1) + \frac{1}{2} \mathbb{P}(\tau_{-1} = n - 1), \quad n \geq 1,
\]

(b) Using reflection symmetry, argue that \( \mathbb{P}(\tau_1 = n - 1) = \mathbb{P}(\tau_{-1} = n - 1) \) for all \( n \geq 1 \); and combine this with (a) to deduce that

\[
\mathbb{P}(\rho_1 = n) = \mathbb{P}(\tau_1 = n - 1) \quad \text{for all } n \geq 1.
\]

Problem #2: Continue with the notation in Problem #1, and, for \( \alpha > 0 \), set \( u_\alpha(k) = \mathbb{E}\left[e^{-\alpha \tau_k}\right] \) for \( k \in Z \).

(a) As in the first part of (b) in Problem #1, argue that \( u_\alpha(k) = u_\alpha(-k) \)

(b) Argue that, for any \( k \geq 1 \) and \( n \geq 1 \), \( \mathbb{P}(\tau_k = n) = 0 \) if \( n < k \), and, in general,

\[
\mathbb{P}(\tau_{k+1} = n) = \sum_{m=1}^{n-1} \mathbb{P}(\tau_1 = m) \mathbb{P}(\tau_k = n - m).
\]

(c) Assuming the result in (b), show that, for each \( k \geq 1 \), \( u_\alpha(k) = u_\alpha(1)u_\alpha(k) \), and conclude that \( u_\alpha(k) = u_\alpha(1)^k \).

(d) By thinking again about what happens on the first step of the random walk, show that

\[
u_{\alpha}(1) = \frac{1}{2} e^{-\alpha} + \frac{1}{2} e^{-\alpha} u_{\alpha}(2).
\]

After combining this with the result in (c), conclude first that \( u_\alpha(1)^2 - 2e^\alpha u_{\alpha}(1) + 1 = 0 \) and then that \( u_{\alpha}(1) = e^\alpha - \sqrt{e^{2\alpha} - 1} \).

(e) From (b) in Problem #1 and (d) above, deduce that

\[
\mathbb{E}\left[e^{-\alpha \rho_1}\right] = 1 - \sqrt{1 - e^{-2\alpha}}.
\]

(f) Starting from (*), show first that

\[
\mathbb{P}(\rho_1 < \infty) = \lim_{\alpha \searrow 0} \mathbb{E}\left[e^{-\alpha \rho_1}\right] = 1,
\]

and second that

\[
\mathbb{E}[\rho_1] = \lim_{\alpha \searrow 0} \mathbb{E}\left[\rho_1 e^{-\alpha \rho_1}\right] = \lim_{\alpha \searrow 0} \frac{d}{d\alpha} \mathbb{E}\left[e^{-\alpha \rho_1}\right] = \infty.
\]

The first of these confirms the conclusion already obtained at the end of §1; the second shows that, although \( \rho_1(\omega) < \infty \) with probability 1, its expected value is nonetheless infinite.