Some Recurrence Criteria

We give several criteria with which to test when a state is recurrent. Throughout, we will think of functions \( u \) on the state space as column vectors. Thus, the transition probability matrix \( P \) acts on the left of functions: \( Pu \) is the function such that \((Pu)_i = \sum_j P_{ij} u_j\).

**Theorem (A).** If \( u \) is a non-negative function which satisfies \( Pu \leq u \) and \((Pu)_j < u_j \), then \( j \) is transient.

**Proof.** Set \( f = u - Pu \), and note that

\[
  u_j \geq u_j - (P^nu)_j = \sum_{m=0}^{n-1} ((P^m u)_j - (P^{m+1} u)_j) = \sum_{m=0}^{n-1} (P^m f)_j \geq f_j \sum_{m=0}^{n-1} (P^m)_{jj}.
\]

Thus \( \mathbb{E}[T_j] = \sum_{n=0}^{\infty} (P^n)_{jj} \leq \frac{\nu_j}{f_j} < \infty \). □

**Lemma 1.** Let \( u \) be a non-negative function and \( S \) a non-empty subset of the state space. If \((Pu)_i \leq u_i \) for all \( i \notin S \) and \( \zeta = \inf\{n \geq 1 : X_n \in S\} \), then

\[
  \mathbb{E}[u_{X_{n \wedge \zeta}} \mid X_0 = i] \leq u_i \quad \text{for all } i.
\]

Moreover, if the inequality in the hypothesis is replaced by equality, the the inequality in the conclusion can be replaced by equality.

**Proof.** Set \( A_n = \{ \zeta > n \} \). Then, by the Markov property, for any \( i \),

\[
  \mathbb{E}[u_{X_{n+1 \wedge \zeta}} \mid X_0 = i] = \mathbb{E}[u_{X_{n+1} \wedge \zeta} 1_{A_n} \mid X_0 = i] + \sum_{k \notin S} \mathbb{E}[u_{X_{n+1} \wedge \zeta} 1_{A_k} | (X_n) \mid X_0 = i] \\
  = \mathbb{E}[u_{X_{n+1} \wedge \zeta} 1_{A_n} \mid X_0 = i] + \sum_{k \notin S} \mathbb{E}[(Pu)_k 1_{A_k} \mid X_n, X_0 = i] \\
  \leq \mathbb{E}[u_{X_{n+1} \wedge \zeta} 1_{A_n} \mid X_0 = i] + \mathbb{E}[u_{X_{n+1}} 1_{A_n} \mid X_0 = i] = \mathbb{E}[u_{X_{n \wedge \zeta}} \mid X_0 = i].
\]

Clearly, the same argument works just as well in the case of equality. □

**Theorem (B).** Let \( j \) be given, and set \( C = \{i : i \mapsto j\} \). If \( j \) is recurrent, then the only bounded functions \( u \) which satisfy \( u_i = (Pu)_i \) for all \( i \in C \setminus \{j\} \) are constant on \( C \). On the other hand, if \( j \) is transient, then the function \( u \) given by

\[
  u_i = \begin{cases} 
    1 & \text{if } i = j \\
    \mathbb{P}(\rho_j < \infty \mid X_0 = i) & \text{if } i \neq j
  \end{cases}
\]

is a bounded, non-constant solution for \( u_i = (Pu)_i \) for all \( i \neq j \).

**Proof.** In proving the first part, we will assume, without loss in generality, that the \( P \) is irreducible and therefore that \( C \) is the whole of the state space. By applying Lemma 1 with \( S = \{j\} \), we see that, for \( i \neq j \),

\[
  u_i = u_j \mathbb{P}(\rho_j \leq n \mid X_0 = i) + \mathbb{E}[u_{X_n} 1_{\rho_j > n} \mid X_0 = i].
\]

Hence, since \( \mathbb{P}(\rho_j < \infty \mid X_0 = i) = 1 \) and \( u \) is bounded, we get \( u_i = u_j \) after letting \( n \to \infty \).

To prove the second part, let \( u \) be given by the above prescription, and begin by observing that, because \( j \) is transient,

\[
  1 > \mathbb{P}(\rho_j < \infty \mid X_0 = j) = P_{jj} + \sum_{i \neq j} P_{ji} u_i \geq P_{jj} + (1 - P_{jj}) \inf_{i \neq j} u_i.
\]

From this, one sees first that \( P_{jj} < 1 \) and then \( \inf_{i \neq j} u_i < 1 = u_j \). That is, \( u \) is bounded and non-constant. At the same time, when \( i \neq j \), by conditioning on what happens at time 1, we know that

\[
  u_i = \mathbb{P}(\rho_j < \infty \mid X_0 = i) = P_{ij} + \sum_{k \neq j} P_{ik} \mathbb{P}(\rho_j < \infty \mid X_0 = k) = (P u)_i.
\]
Theorem (C). Let \( \{B_m : m \geq 0\} \) be a non-decreasing sequence of subsets of the state space with the property that

\[
\mathbb{P}(\exists n \in \mathbb{N} \ X_n \notin B_m \ | \ X_0 = j) = 1 \quad \text{for some } j \in B_0 \text{ and all } m \geq 0.
\]

If there exists a non-negative solution \( u \) to \((Pu)_i \leq u_i, \ i \neq j\), for which \( a_m \equiv \inf_{i \notin B_m} u_i \to \infty \) as \( m \to \infty \), then \( j \) is recurrent.

Proof. For each \( m \geq 0 \), set \( S_m = \{j\} \cup B_m^C \), and take \( \zeta_m = \inf\{n \geq 1 : X_n \in S_m\} = \rho_j \wedge \tau_m \), where \( \tau_m \equiv \inf\{n \geq 1 : X_n \notin B_m\} \). By Lemma 3,

\[
u_j \geq \mathbb{E}[u_{X_n \wedge \zeta_m} \ | \ X_0 = j] \geq a_m \mathbb{P}(\tau_m \leq n \wedge \rho_j \ | \ X_0 = j)
\]

for all \( n \geq 0 \). Hence, because \( \mathbb{P}(\tau_m < \infty \ | \ X_0 = j) = 1 \), we conclude, after letting \( n \to \infty \), that \( u_j \geq a_m \mathbb{P}(\tau_m \leq \rho_j \ | \ X_0 = j) \) for all \( m \geq 0 \). Thus, \( \lim_{m \to \infty} \mathbb{P}(\tau_m \leq \rho_j \ | \ X_0 = j) = 0 \). But this means that

\[
\mathbb{P}(\rho_j < \infty \ | \ X_0 = j) \geq \mathbb{P}(\rho_j < \tau_m \ | \ X_0 = j) = 1 - \mathbb{P}(\tau_m \leq \rho_j \ | \ X_0 = j) > 1,
\]

and so \( j \) is recurrent. \( \Box \)

Corollary. Assume that \( \mathbf{P} \) is irreducible, and let \( \{F_m : m \geq 0\} \) be a non-decreasing sequence of finite subsets of the state space. If \( j \in F_0 \) and there exists \( u \) is a non-negative solution to \((Pu)_i \leq u_i, \ i \neq j\) with \( \inf_{i \notin F_m} u_i \to \infty \), then \( j \) is recurrent.

Proof. In view of Theorem (C), it suffices for us to check that \( \mathbb{P}(\exists n \in \mathbb{N} \ X_n \notin F_m \ | \ X_0 = j) = 1 \) for all \( m \geq 0 \). To this end, let \( \tau_m = \inf\{n \geq 1 : X_n \notin F_m\} \). By irreducibility, \( \mathbb{P}(\tau_m < \infty \ | \ X_0 = i) > 0 \) for all \( m \) and \( i \). Hence, because \( F_m \) is finite, for each \( m \) there exists a \( \theta_m \in (0, 1) \) and \( N_m \geq 1 \) such that \( \mathbb{P}(\tau_m > N_m \ | \ X_0 = i) \leq \theta_m \) for all \( i \in F_m \). But this means that

\[
\mathbb{P}(\tau_m > (\ell + 1)N_m \ | \ X_0 = j) = \sum_{i \in F_m} \mathbb{P}(\tau_m > (\ell + 1)N_m \ & \ X_{\ell N_m} = i \ | \ X_0 = j)
\]

\[
= \sum_{i \in F_m} \mathbb{P}(\tau_m > N_m \ | \ X_0 = i) \mathbb{P}(\tau_m > \ell N_m \ & \ X_{\ell N_m} = i \ | \ X_0 = j) \leq \theta_m \mathbb{P}(\tau_m > \ell N_m \ | \ X_0 = j).
\]

Thus, \( \mathbb{P}(\tau_m > \ell N_m \ | \ X_0 = j) \leq \theta_m^\ell \), and so \( \mathbb{P}(\tau_m = \infty \ | \ X_0 = j) = 0 \). \( \Box \)