1.1.11: Given $C$ and $\epsilon > 0$, set

$$C(\epsilon) = \left\{ I \in C : \sup_I f - \inf_I f \geq \epsilon \right\}.$$  

Assume that $f$ is Riemann integrable, and let $\epsilon > 0$ be given. Then there exists a $\delta > 0$ such that

$$U(f; C) - L(f; C) < \epsilon^2$$

whenever $\|C\| < \delta$. Thus, if $\|C\| < \delta$, then

$$\epsilon \sum_{I \in C(\epsilon)} \text{vol}(I) \leq \sum_{I \in C(\epsilon)} \left( \sup_I f - \inf_I f \right) \text{vol}(I)$$

$$\leq \sum_{I \in C} \left( \sup_I f - \inf_I f \right) \text{vol}(I) = U(f; C) - L(f; C) < \epsilon^2.$$  

Conversely, suppose that for each $\epsilon > 0$ there is a $C(\epsilon)$ for which

$$\sum_{I \in C(\epsilon)} \text{vol}(I) < \epsilon.$$  

Then

$$U(f; C(\epsilon)) - L(f; C(\epsilon))$$

$$= \sum_{I \in C(\epsilon)} \left( \sup_I f - \inf_I f \right) \text{vol}(I) + \sum_{I \in C \setminus C(\epsilon)} \left( \sup_I f - \inf_I f \right) \text{vol}(I)$$

$$\leq 2\|f\|_u \sum_{I \in C(\epsilon)} \text{vol}(I) + \epsilon \sum_{I \in C \setminus C(\epsilon)} \text{vol}(I)$$

$$\leq 2\|f\|_u \epsilon + |J| = \epsilon (2\|f\|_u + |J|) \longrightarrow 0 \quad \text{as} \quad \epsilon \searrow 0.$$  

Hence, $\inf_C U(f; C) = \sup_C L(f; C)$, and so $f$ is Riemann integrable.

1.2.17: (i) Given $\epsilon > 0$, choose $\delta > 0$ so that $|\psi'(y) - \psi'(x)| < \epsilon$ whenever $|y - x| < \delta$. Next, given $C$ with $\|C\| < \delta$, use the Mean Value Theorem to choose $\eta(I) \in \hat{I}$ for each $I \in C$ so that $\Delta_I \psi = \psi'(\eta(I)) \text{vol}(I)$. Then, for any $\xi \in \Xi(C)$,

$$|\mathcal{R}(\varphi | \psi; C, \xi) - \mathcal{R}(\varphi \psi'; C, \xi)|$$

$$\leq \sum_{I \in C} |\varphi(\xi(I))||\psi'(\eta(I)) - \psi'(\xi(I))| \leq \epsilon \|\varphi\|_u \text{vol}(J).$$  

Hence, since $\varphi \psi'$ is continuous and therefore Riemann integrable, we have now proved that

$$\lim_{\|C\| \to 0} \sup_{\xi \in \Xi(C)} \left| \mathcal{R}(\varphi | \psi; C, \xi) - (R) \int_J \varphi(x) \psi'(x) \, dx \right| = 0.$$
(ii) Given \( \epsilon > 0 \), choose \( 0 < \delta < \min_{1 \leq m \leq n} (a_m - a_{m-1}) \) so that \( |\varphi(y) - \varphi(x)| < \epsilon \) whenever \( |y - x| < \delta \). Next, let \( \mathcal{C} \) with \( ||\mathcal{C}|| < \delta \) be given. Clearly, no \( I \in \mathcal{C} \) contains more than one of the \( a_m \)'s and each \( a_m \) is in at most two \( I \)'s. In fact, for any \( I \in \mathcal{C} \),

\[
\Delta_I \psi = \begin{cases} 
0 & \text{if } I \cap \{a_0, \ldots, a_n\} = \emptyset \\
d_m & \text{if } a_m \in I \\
d_m^+ = \psi(a_m^+) - \psi(a_m) & \text{if } I^- = a_m \\
d_m^- = \psi(a_m) - \psi(a_m^-) & \text{if } I^+ = a_m.
\end{cases}
\]

Thus, if \( M_0 = \{m : a_m \in \hat{I} \text{ for some } I \in \mathcal{C} \} \)

and

\( M_{\pm} = \{m : a_m = I^\pm \text{ for some } I \in \mathcal{C} \} \),

and if \( \xi_0 \in \mathcal{C} \) is chosen so that \( a_m \in I \implies \xi_0(I) = a_m \), then

\[
\mathcal{R}(\varphi \mid \psi; \mathcal{C}, \xi_0) = \sum_{m \in M_0} \varphi(a_m) d_m + \sum_{m \in M^+} \varphi(a_m) d_m^+ + \sum_{m \in M^-} \varphi(a_m) d_m^-
\]

\[
= \sum_{m=0}^n \varphi(a_m) d_m,
\]

since \( (\text{when } d_m^0 = d_m^\pm \equiv 0) d_m = d_m^- + d_m^+, 0 \leq m \leq n \). At the same time, for any \( \xi \in \Xi(\mathcal{C}) \),

\[
|\mathcal{R}(\varphi \mid \psi; \mathcal{C}, \xi) - \mathcal{R}(\varphi \mid \psi; \mathcal{C}, \xi_0)| \leq \epsilon \sum_{0}^{n} (|d_m^-| + |d_m^+|).
\]

Hence, we have now shown that

\[
\lim_{||\mathcal{C}|| \to 0} \sup_{\xi \in \Xi(\mathcal{C})} \left| \mathcal{R}(\varphi \mid \psi; \mathcal{C}, \xi) - \sum_{m=0}^{n} \varphi(a_m) d_m \right| = 0.
\]

(iii) Without loss of generality, assume that \( \alpha = \beta = 1 \). Given \( \epsilon > 0 \), choose \( \delta > 0 \) so that

\[
||\mathcal{C}|| < \delta \implies \max_{i \in \{1, 2\}} \sup_{\xi \in \Xi(\mathcal{C})} \left| \mathcal{R}(\varphi_i \mid \psi; \mathcal{C}, \xi) - (R) \int_{J} \varphi_i(x) d\psi(x) \right| < \epsilon.
\]

Because \( \mathcal{R}(\varphi_1 + \varphi_2 \mid \psi; \mathcal{C}, \xi) = \mathcal{R}(\varphi_1 \mid \psi; \mathcal{C}, \xi) + \mathcal{R}(\varphi_2 \mid \psi; \mathcal{C}, \xi) \), it is then clear that

\[
\sup_{\xi \in \Xi(\mathcal{C})} \left| \mathcal{R}(\varphi_1 + \varphi_2 \mid \psi; \mathcal{C}, \xi) - (R) \int_{J} \varphi_1(x) d\psi(x) \right| < 2\epsilon
\]

whenever \( ||\mathcal{C}|| < \delta \).
(iv) We begin by showing that $\varphi$ is $\psi$-Riemann integrable on both $J_1$ and $J_2$. To this end, let $\epsilon > 0$ be given, and choose $\delta > 0$ so that

\[ (*) \quad \|C\| < \delta \implies \sup_{\xi \in \Xi(C)} \left| \mathcal{R}(\varphi | \psi; C, \xi) - (R) \int_{J_1} \varphi(x) \, d\psi(x) \right| < \epsilon. \]

Now, suppose that $C_1$ and $C'_1$ are finite, non-overlapping covers of $J_1$ with mesh size less than $\delta$. Let $C_2$ be any finite, non-overlapping cover of $J_2$ with mesh size less than $\delta$, and set $C = C_1 \cup C_2$ and $C' = C'_1 \cup C_2$. Next, let $\xi_1 \in \Xi(C_1)$ and $\xi'_1 \in \Xi(C'_1)$ be given, and define $\xi \in \Xi(C)$ and $\xi' \in \Xi(C')$ so that

\[ \xi(I) = \begin{cases} \xi_1(I) & \text{if } I \in C_1 \\ \xi_2(I) & \text{if } I \in C_2 \end{cases} \quad \text{and} \quad \xi'(I) = \begin{cases} \xi'_1(I) & \text{if } I \in C'_1 \\ \xi_2(I) & \text{if } I \in C_2, \end{cases} \]

where $\xi_2 \in \Xi(C_2)$. Then, by $(*),$ \n\[
\left| \mathcal{R}(\varphi | \psi; C, \xi) - \mathcal{R}(\varphi | \psi; C', \xi') \right| < 2\epsilon.
\]

At the same time,

\[ \mathcal{R}(\varphi | \psi; C_1, \xi_1) - \mathcal{R}(\varphi | \psi; C'_1, \xi'_1) = \mathcal{R}(\varphi | \psi; C, \xi) - \mathcal{R}(\varphi | \psi; C', \xi'). \]

Thus, by Cauchy’s convergence criterion, we have now shown that $\varphi$ is $\psi$-Riemann integrable on $J_1$. The same argument shows that $\varphi$ is $\psi$-Riemann integrable on $J_2$.

To prove the asserted equality, let $\epsilon > 0$ be given, and choose $\delta > 0$ so that $(* \text{ holds})$. Next, choose $C_i$ for $J_i$ and $\xi_i \in \Xi(C_i)$ so that $\|C_i\| < \delta$ and

\[ (**) \quad \max_{i \in \{1, 2\}} \left| \mathcal{R}(\varphi | \psi; C_i, \xi_i) - (R) \int_{J_i} \varphi(x) \, d\psi(x) \right| < \epsilon. \]

Finally, set $C = C_1 \cup C_2$, and define $\xi \in \Xi(C)$ by

\[ \xi(I) = \begin{cases} \xi_1(I) & \text{if } I \in C_1 \\ \xi_2(I) & \text{if } I \in C_2. \end{cases} \]

Then,

\[ \mathcal{R}(\varphi | \psi; C, \xi) = \mathcal{R}(\varphi | \psi; C_1, \xi_1) + \mathcal{R}(\varphi | \psi; C_2, \xi_2), \]

and so $(* \text{ and } (**) \text{ imply that}$

\[ \left| (R) \int_{J_1} \varphi(x) \, d\psi(x) - (R) \int_{J_2} \varphi(x) \, d\psi(x) - (R) \int_{J_1} \varphi(x) \, d\psi(x) \right| < 3\epsilon. \]

1.2.21: Let $\psi \in C(J; \mathbb{R})$. What we must show is that, for each $C$ and $\epsilon > 0$, there is a $\delta > 0$ such that

\[ S(\psi; C) - S(\psi; C') < \epsilon \]
whenever \( \|C\| < \delta \). To this end, suppose that
\[
C = \left\{ [a_0, a_1], \ldots, [a_{n-1}, a_n] \right\}
\]
where \( J^- = a_0 < \cdots < a_n = J^+ \). Next, given \( \epsilon > 0 \), choose \( 0 < \delta < \min_{1 \leq m \leq n} (a_m - a_{m-1}) \) so that
\[
\omega_\delta(\delta) \equiv \sup \left\{ |\psi(t) - \psi(s)| : s, t \in J \text{ and } |t - s| < \delta \right\} < \frac{\epsilon}{2n}.
\]
If \( \|C\| < \delta \) and \( A \) is the set of those \( I' \in C' \) for which there is an \( I \in C \) with \( I' \subseteq I \), then for each \( I' \in B \equiv C' \setminus A \), there is precisely one \( m \in \{1, \ldots, n-1\} \) for which \( a_m \in I' \). In particular, because \( C' \) is non-overlapping, \( B \) has at most \( n \) elements. Moreover, if \( I' \in B \), \( a_m \in I' \), and we use \( L(I') \) and \( R(I') \) to denote \( I' \cap [a_{m-1}, a_m] \) and \( I' \cap [a_m, a_{m+1}] \), respectively, then
\[
C \cup C' = A \cup \bigcup_{I' \in B} \left\{ [L(I'), R(I')] \right\}.
\]
Hence,
\[
S(\psi; C) - S(\psi; C') \leq S(\psi; C' \cup C) - S(\psi; C) \leq \sum_{I' \in B} \left( |\Delta_{L(I')}\psi| + |\Delta_{R(I')}\psi| \right) \leq 2n\omega_\psi(\delta) < \epsilon.
\]
Finally, by (1.2.10) and (1.2.11), the analogous result for \( \text{Var}_- \) and \( \text{Var}_+ \) follow immediately.

Now, suppose that \( \psi \in C^1(J; \mathbb{R}) \). For \( n \in \mathbb{Z}^+ \) and \( 1 \leq m < n \), set \( a_{m,n} = J^- + \frac{m}{n} \Delta J \), and use the Mean Value Theorem to choose \( \xi_{m,n} \in (a_{m,n}, a_{m+1,n}) \) so that \( \varphi(a_{m+1,n}) - \varphi(a_{m,n}) = \varphi'(\xi_{m,n}) \frac{\Delta J}{n} \). Then
\[
\text{Var}_+(\psi) = \lim_{n \to \infty} \frac{\Delta J}{n} \sum_{m=0}^{n-1} \psi'(\xi_{m,n})^+ = (R) \int_J \varphi'(t)^+ dt,
\]
and similarly for \( \text{Var}_-(\psi) \) and \( \text{Var}(\psi) \).

1.2.24:

(i) Because \( \sqrt{\alpha^2 + \beta^2} \leq \alpha + \beta \) for all \( \alpha, \beta \geq 0 \),
\[
\sum_{I \in C} \sqrt{\text{vol}(I)^2 + (\Delta_I \psi)^2} \leq \sum_{I \in C} (\text{vol}(I) + |\Delta_I \psi|) \leq (d - c) + \text{Var}(\psi; [c, d]),
\]
which proves the upper bound. The lower bound is an application of the classical triangle inequality. Namely, for any \( n \geq 1 \) and \( x_1, \ldots, x_n \in \mathbb{R}^2 \), \( \sum_{m=1}^n x_m | \leq \sum_{m=1}^n |x_m| \). Equivalently, for any \( \{ (a_m, b_m) : 1 \leq m \leq n \} \subseteq \mathbb{R}^2 \),
\[
\sum_{m=1}^n \sqrt{a_m^2 + b_m^2} \geq \left( \sum_{m=1}^n |a_m| \right) + \left( \sum_{m=1}^n |b_m| \right).
\]
Hence,
\[
\text{Arc}(\psi; [c, d]) \geq \sum_{I \in C} \sqrt{\text{vol}(I)^2 + (\Delta_I \psi)^2} \geq (d - c)^2 + \left( \sum_{I \in C} |\Delta_I \psi| \right)^2,
\]
from which the desired estimate follows when one takes the supremum over \( C \).
(ii) When $\psi(x) = ax + b$, it is obvious that $\text{Arc}(\psi; [c, d]) = (d-c)\sqrt{1+a^2} = \sqrt{(d-c)^2 + \text{Var}(\psi; J)^2}$. To show that the upper is achieved when $\psi$ is a pure jump function, for each $\epsilon > 0$, define $D_\epsilon$ and choose $\{C_n : n \geq 1\}$ as in the solution to (v) in Exercise 1.2.23. Then

$$\text{Arc}(\psi; J) \geq \sum_{I \in C_n \setminus D_\epsilon \neq \emptyset} \text{vol}(I) + \sum_{I \in C_n \cap D_\epsilon = \emptyset} |\Delta_I\psi|$$

$$\rightarrow (d-c) + \sum_{x \in D_\epsilon} |\psi(x) - \psi(x^-)| \quad \text{as } n \to \infty.$$ 

Hence, after letting $\epsilon \searrow 0$, the result follows.

(iii) The solution to this part is essentially the same as the solution to Exercise 1.2.22. To see this, given a function $\Psi : J \to \mathbb{R}^2$, define $\text{Var}(\Psi; J) = \sup_{C} \sum_{I \in C} |\Delta_I\Psi|$, where the supremum is over finite, non-overlapping, exact covers of $J$ by intervals, and $\Delta_I\Psi = \Psi(I^+) - \Psi(I^-)$. Just as in the solution to Exercise 1.2.22, when $\Psi \in C(J; \mathbb{R}^2)$,

$$\text{Var}(\Psi; J) = \lim_{\|C\| \to 0} \sum_{I \in C} |\Delta_I\Psi|.$$ 

Now set $\Psi(x) = (x, \psi(x))$, observe that $\text{Var}(\Psi; J) = \text{Arc}(\psi; J)$, and apply the preceding.

1.3.19: The first step is to note that, for any $a < b$, $L > 0$, and $\varphi \in C([aL, bL]; \mathbb{R})$,

$$\left(\text{R}\right) \int_{[aL, bL]} \varphi(x) \, dx = L(R) \int_{[a, b]} \varphi(Lx) \, dx.$$ 

Hence,

$$\left(\text{R}\right) \int_{[0, n]} f(x) \, dx = n(R) \int_{[0, 1]} f(nx) \, dx.$$ 

Now apply (1.3.15), with $f$ replaced by $f(n \cdot)$, to see that

$$\left(\text{R}\right) \int_{[0, n]} f(x) \, dx - \sum_{m=1}^{n} f(m) + \sum_{k=1}^{\ell} b_k (f^{(k-1)}(n) - f^{(k-1)}(0))$$

$$= \sum_{k=1}^{\ell} b_{k-n} n^{k+1} \Delta_n^{(k)} f^{(\ell)}(n \cdot).$$ 

By another application of (*), $n^{k+1} \Delta_n^{(k)} f^{(\ell)}(n \cdot)$ equals

$$\frac{n}{k!} \sum_{m=1}^{n} \left( \int_{m, m-1} (nx - (m-1))^{k} \left( f^{(\ell)}(nx) - f^{(\ell)}(m) \right) \, dx \right)$$

$$= \frac{1}{k!} \sum_{m=1}^{n} \left( \int_{m-1, m} (x - (m-1))^{k} \left( f^{(\ell)}(x) - f^{(\ell)}(m) \right) \, dx \right),$$

and from this the result is clear.
1.3.21: To prove the uniqueness assertion, note that $P_\ell$ must be an \( \ell \)th order polynomial, and write 

\[
P_\ell(x) = \sum_{k=0}^{\ell} a_{k,\ell} x^k.
\]

Then (1.3.22) implies that 

\[
a_{0,0} = 1, \quad a_{k,\ell} = -\frac{1}{k} a_{k-1,\ell-1} \quad \text{for} \quad 1 \leq k \leq \ell, \quad \text{and} \quad a_{1,\ell} = -\sum_{k=2}^{\ell} a_{k,\ell} \quad \text{for} \quad \ell \geq 2.
\]

By combining the last two of these, one sees that, for \( \ell \geq 1 \),

\[
a_{0,\ell} = -a_{1,\ell+1} = \sum_{k=2}^{\ell+1} a_{k,\ell+1} = -\sum_{k=1}^{\ell} \frac{a_{k,\ell}}{k+1}.
\]

Hence, working by induction on \( \ell \), one can easily check that the \( \{a_{k,\ell} : 1 \leq k \leq \ell\} \) is uniquely determined for all \( \ell \geq 0 \). To show that the \( P_\ell \)'s in (1.3.20) have the properties in (1.3.22), first note that \( P_0 \equiv 1 \) is trivial. Second, 

\[
P_\ell'(x) = \sum_{k=1}^{\ell+1} \frac{(-1)^k b_{\ell+1-k}}{(k-1)!} x^{k-1} = -\sum_{k=0}^{\ell} \frac{(-1)^k b_{\ell-k-1}}{k!} x^{k} = -P_\ell(x).
\]

Finally, if \( \ell \geq 2 \),

\[
P_\ell(1) = \sum_{k=1}^{\ell} \frac{(-1)^k b_{\ell-k}}{k!} = b_{\ell} - b_{\ell-1} + \sum_{k=2}^{\ell} \frac{(-1)^k b_{\ell-k}}{k!}
\]

\[
= b_{\ell} - b_{\ell-1} + \sum_{k=0}^{\ell-2} \frac{(-1)^k b_{\ell-2-k}}{(k+2)!} = b_{\ell} = P_\ell(0),
\]

where the second to last equality comes from (1.3.7).

1.3.23: As suggested, consider the generating function \( B(\lambda) \) in (1.3.12). If we show that \( B'(\lambda) = B'(-\lambda) \), then we will know that \( (-1)^k b_k = b_k \) for \( k \geq 2 \) and therefore that \( b_{2k+1} = 0 \) for all \( k \geq 1 \). To show that \( B'(\lambda) = B'(-\lambda) \), note that

\[
B(\lambda) = -\frac{1}{\lambda} + \frac{e^\lambda}{e^\lambda - 1} \quad \text{and} \quad B(-\lambda) = \frac{1}{\lambda} - \frac{1}{e^\lambda - 1}
\]

and therefore that \( B(\lambda) + B(-\lambda) = 1 \) for small \( \lambda \neq 0 \). Clearly this implies that \( B'(\lambda) - B'(-\lambda) = 0 \).