Hodge Theory

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Contents

1 Definition and properties of $\star$.

Let V be an n-dimensional vector space over $\mathbf{R}$ and B a bilinear form on V. B induces a bilinear form on $\wedge^p V$, also denoted by B determined by its value on decomposable elements as

$$B(\mu,\nu) := \det(B(u_i,v_j)), \quad \mu = u_1 \wedge \cdots u_p, \quad \nu = v_1 \wedge \cdots v_p.$$

Suppose we also have fixed an element $\Omega \in \wedge^n V$ which identifies $\wedge^n V$ with $\mathbf{R}$. Exterior multiplication then identifies $\wedge^{n-p} V$ with $(\wedge^p V)^*$ and B maps $\wedge^p V \to (\wedge^p V)^*$. We thus get a composite map

$$\star : \wedge^p V \to \wedge^{n-p} V$$

characterized by

$$\alpha \wedge \star\beta = B(\alpha, \beta)\Omega. \tag{1}$$

Properties of $\star$.

- **Dependence on Ω.** If $\Omega_1 = \lambda\Omega$ then

  $$\star_1 = \lambda\star$$

  as follows immediately from the definition.

- **Dependence on B.** Suppose that

  $$B_1(v, w) = B(v, Jw), \qquad J \in \text{End } V.$$

  Extend J to an element of $\wedge V$ by $J(v_1 \wedge \cdots v_p) := Jv_1 \wedge \cdots \wedge Jv_p$. Thus the extended bilinear forms are also related by

  $$B_1(\mu, \nu) = B(\mu, J\nu)$$

  and hence

  $$\star_1 = \star \circ J.$$

- **Behavior under direct sums.** Suppose

  $$V = V_1 \oplus V_2,\ B = B_1 \oplus B_2,\ \Omega = \Omega_1 \wedge \Omega_2$$

  under the identification

  $$\wedge V = \wedge V_1 \otimes \wedge V_2.$$

  Then for $\alpha_1, \beta_1 \in \wedge^r V_1,\ \alpha_1, \beta_2 \in \wedge^s V_2$ we have

  $$B(\alpha_1 \wedge \alpha_2, \beta_1 \wedge \beta_2) = B(\alpha_1, \beta_1)B(\alpha_2, \beta_2)$$

  and

  $$(\alpha_1 \wedge \alpha_2) \wedge \star(\beta_1 \wedge \beta_2) = B(\alpha_1 \wedge \alpha_2, \beta_1 \wedge \beta_2)\Omega = B(\alpha_1, \beta_1)\Omega_1 \wedge B(\alpha_2, \beta_2)\Omega_2$$

  while

  $$\alpha_1 \wedge \star_1\beta_1 = B(\alpha_1, \beta_1)\Omega_1, \quad \alpha_2 \wedge \star_2\beta_1 = B(\alpha_2, \beta_2)\Omega_2.$$

  Hence

  $$\star(\omega_1 \wedge \omega_2) = (-1)^{n_1-r)s} \star_1 \omega_1 \wedge \star_2\omega_2 \quad \text{for} \ \ \omega_1 \in \wedge^r V_1,\ \omega_2 \in \wedge^s V_2.$$

  Since $\star_1\omega_1 \wedge \star_2\omega_2 = (-1)^{(n_1-r)(n_2-s)} \star 2\omega_2 \wedge \star_1\omega_1$ we can rewrite the preceding equation as

  $$\star(\omega_1 \wedge \omega_2) = (-1)^{(n_1-r)n_2} \star_2 \omega_2 \wedge \star_1\omega_1.$$

In particular, if n_2 is even we get the simpler looking formula

$$\star(\omega_1 \wedge \omega_2) = \star_2\omega_2 \wedge \star_1\omega_1.$$

So, by induction, if

$$V = V_1 \oplus \cdots \oplus V_m$$

is a direct sum of even dimensional subspaces and $\Omega = \Omega_1 \wedge \cdots \wedge \Omega_m$ then

$$\star(\omega_1 \wedge \cdots \wedge \omega_m) = \omega_m \wedge \cdots \wedge \omega_1, \quad \omega_i \in \wedge(V_i). \tag{2}$$

2 Exterior and interior multiplication.

Suppose that B is non-degenerate. For $u \in V$ we let $e_u : \wedge V \to \wedge V$ denote exterior multiplication by u. For $\gamma \in V^*$ we let $i_\gamma : \wedge V \to \wedge V$ denote interior multiplication by γ. We can also consider the transposes of these operators with respect to B:

$$e_v^\dagger : \wedge^p V \to \wedge^{p-1} V,$$

defined by

$$B(e_v\alpha, \beta) = B(\alpha, e_v^\dagger\beta), \quad \alpha \in \wedge^{p-1}V,\ \beta \in \wedge^p V$$

and

$$i_\gamma^\dagger : \wedge^{p-1}V \to \wedge^p V$$

defined by

$$B(i_\gamma\alpha, \beta) = B(\alpha, i_\gamma^\dagger\beta), \quad \alpha \in \wedge^{p+1}V,\ \beta \in \wedge^p V.$$

We claim that

$$e_v^\dagger = (-1)^{p-1} \star^{-1} e_v \star \tag{3}$$

and

$$i_\gamma^\dagger = (-1)^p \star^{-1} i_\gamma \star \tag{4}$$

on $\wedge^p V$.

Proof of (3). For $\alpha \in \wedge^{p-1}V,\ \beta \in \wedge^p V$ we have

$$\begin{aligned} B(e_v \wedge \alpha, \beta)\Omega &= e_v\alpha \wedge \star\beta \\ &= (-1)^{p-1}\alpha \wedge v \wedge \star\beta \\ &= (-1)^{p-1}\alpha \wedge \star\star^{-1} e_v \star \beta \\ &= (-1)^{p-1} B(\alpha, \star^{-1} e_v \star \beta)\Omega. \quad \square \end{aligned}$$

Proof of (4). Let $\alpha \in \wedge^{p+1}V,\ \beta \in \wedge^p V$ so that

$$\alpha \wedge \star\beta = 0.$$

We have

$$\begin{aligned}
0 &= i_\gamma(\alpha \wedge \star\beta) \\
&= (i_\gamma\alpha) \wedge \star\beta + (-1)^{p-1}\alpha \wedge i_\gamma \star \beta \\
&= (i_\gamma\alpha) \wedge \star\beta + (-1)^{p-1}\alpha \wedge \star(\star^{-1} i_\gamma \star)\beta \quad \text{so} \\
B(i_\gamma\alpha, \beta)\Omega &= (-1)^p B(\alpha, \star^{-1} i_\gamma \star \beta)\Omega. \quad \square
\end{aligned}$$

There are alternative formulas for $e_v^\dagger$ and $i_\gamma^\dagger$ which are useful, and involve dualities between V and V^* induced by B. We let $\langle\ ,\ \rangle$ denote the pairing of V and V^*, so

$$\langle v, \ell \rangle$$

denotes the value of the linear function, $\ell \in V^*$ on $v \in V$. Define the maps

$$L = L_B, \quad \text{and} \quad L^{op} = L_B^{op} : V \to V^*$$

by

$$\langle v, Lw \rangle = B(v, w), \qquad \langle v, L^{op}w \rangle = B(w, v), \qquad v, w \in V. \tag{5}$$

We claim that

$$e_v^\dagger = i_{L^{op}v} \tag{6}$$

$$i_{Lv}^\dagger = e_v \tag{7}$$

Proof. We may suppose that $v \neq 0$ and extend it to a basis $v_1, \ldots, v_n$ of V, with $v_1 = v$. Let $w_1, \ldots, w_n$ be the basis of V determined by

$$B(v_i, w_j) = \delta_{ij}.$$

Let $\gamma^1, \ldots, \gamma^n$ be the basis of V^* dual to $w_1, \ldots, w_n$ and set $\gamma := \gamma_1$. Then

$$\begin{aligned}
\langle w_i, L^{op}v \rangle &= B(v, w_i) \\
&= \delta_{1i} \\
&= \langle w_i, \gamma \rangle \quad \text{so} \\
\gamma &= L^{op}v.
\end{aligned}$$

If $J = (j_1, \ldots, j_p)$ and $K = (k_1, \ldots, k_{p+1}$ are (increasing) multi-indices then

$$B(e_v v^J, w^K) = 0$$

unless $k_1 = 1$ and $k_{r+1} = i_r$, $\quad r = 1, \ldots, p$, in which case

$$B(e_v v^J, w^K) = 1.$$

The same is true for

$$B(v^J, i_\gamma w^K).$$

Hence

$$e_v^\dagger = i_\gamma$$

which is the content of (6).

Similarly, let $w = w_1$ and $\beta = L(w)$ so that

$$i_\beta v_j = B(v_j, w_1) = \delta_{1j}.$$

Then

$$B(i_\beta(v^K), w^J) = 0$$

unless $k_1 = 1$ and $k_{r+1} = j_r, \quad r = 1, \ldots, p$ in which case

$$B(i_\beta(v^K), w^J) = 1$$

and the same holds for $B(v^K, w \wedge w^J)$. This proves (7).

Combining (3) and (6) gives

$$\star^{-1} e_v \star = (-1)^{p-1} i_{L^{op} v}, \tag{8}$$

while combining (4) and (7) gives

$$\star^{-1} i_{Lv} \star = (-1)^p e_v. \tag{9}$$

On any vector space, independent of any choice of bilinear form we always have the identity

$$i_\gamma e_w + e_w i_\gamma = \langle w, \gamma \rangle, \quad v \in V, \ \gamma \in V^*.$$

If $\gamma = L^{op} v$, then $\langle w, \gamma \rangle = B(v, w)$ so (3) implies

$$e_v^\dagger e_w + e_w e_v^\dagger = B(v, w) I. \tag{10}$$

3 The case of B symmetric positive definite.

In this case it is usual to choose Ω such that $\|\Omega\| = 1$. The only choice left is then of an orientation. Suppose we have fixed an orientation and so a choice of Ω. To compute $\star$ it is enough to compute it on decomposable elements. So let U be a p-dimensional subspace of V and $u_1 \wedge \cdots \wedge w_p$ an orthonormal basis of U. Let W be the orthogonal complement of U and let $w_1, \ldots, w_q$ be an orthonormal basis of W where $q := n - p$. Then

$$u_1 \wedge \cdots \wedge u_p \wedge w_1 \wedge \cdots \wedge w_q = \pm \Omega.$$

We claim that

$$\star(u_1 \wedge \cdots \wedge u_p) = \pm w_1 \wedge \cdots \wedge w_q.$$

We need only check that

$$B(\alpha, u_1 \wedge \cdots \wedge u_p)\Omega = \pm \alpha \wedge w_1 \wedge \cdots \wedge w_q$$

for $\alpha \in \wedge^p V$ which are wedge products of u_i and w_j since $u_1, \ldots, u_p, w_1, \ldots, w_q$ form a basis of V. Now if any w's are involved in this product decomposition both sides vanish. And if $\alpha =_1 \wedge \cdots \wedge u_p$ then this is the definition of the $\pm$ occurring in the formula.

Suppose we have chosen both bases so that $\pm = +$. Then

$$\star(u_1 \wedge \cdots \wedge u_p) = w_1 \wedge \cdots \wedge w_q$$

while

$$\star(w_1 \wedge \cdots w_q) = \pm u_1 \wedge \cdots u_p$$

where $\pm$ is the sign of the permutation involved in moving all the w's past the u's. This sign is $(-1)^{p(n-p)}$. We conclude

$$\star^2 = (-1)^{p(n-p)} \qquad \text{on} \quad \wedge^p V. \tag{11}$$

In particular

$$\star^2 = (-1)^p \qquad \text{on} \quad \wedge^p V \quad \text{if } n \text{ is even.} \tag{12}$$

4 The case of B symplectic.

Suppose $n = 2m$ and $e_1, \ldots, e_m, f_1, \ldots, f_m$ is a basis of V with

$$B(e_i, f_j) = \delta_{ij}, \quad B(e_i, e_j) = B(f_i, f_j) = 0.$$

We take

$$\Omega := e_1 \wedge f_1 \wedge e_2 \wedge f_2 \cdots \wedge e_m \wedge f_m$$

which is clearly independent of the choice of basis with the above properties. If we let V_i denote the two dimensional space spanned by e_i, f_i with B_i the restriction of B to V_i and $\Omega_i := e_i \wedge f_i$ then we are in the direct sum situation and so can apply (2).

So to compute $\star$ in the symplectic situation it is enough to compute it for a two dimensional vector space with basis e, f satisfying

$$B(e, f) = 1, \quad e \wedge f = \Omega.$$

Now

$$B(e, e) = 0 = e \wedge e, \quad B(f, e)\Omega = -\Omega = f \wedge e$$

so

$$\star e = e.$$

Similarly

$$\star f = f.$$

On any vector space the "induced bilinear form" on $\wedge^0$ is given by

$$B(1, 1) = 1$$

so

$$\star 1 = \Omega.$$

On the other hand,

$$B(e \wedge f, e \wedge f) = \det \begin{pmatrix} B(e,e) & B(e,f) \\ B(f,e) & B(f,f) \end{pmatrix} = \det \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 1.$$

So

$$\star(e \wedge f) = 1.$$

This computes $\star$ is all cases for a two dimensional symplectic vector space. We conclude that

$$\star^2 = \mathrm{id} \tag{13}$$

first for a two dimensional symplectic vector space and then, from (2), for all symplectic vector spaces.

5 Graded $sl(2)$.

We consider the three dimensional graded Lie algebra

$$g = g_{-2} \oplus g_0 \oplus g_2$$

where each summand is one dimensional with basis F, H, E respectively and bracket relations

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H.$$

For example, $g = sl(2)$ with

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let V be a symplectic vector space with symplectic form, B and symplectic basis

$$u_1, \dots, u_m, v_1, \dots, v_m$$

so

$$B(u_i, u_j) = 0, = B(v_i, v_j), \quad B(u_i, v_j) = \delta_{ij}.$$

Let

$$\omega := u_1 \wedge v_1 + \cdots + u_m \wedge v_m.$$

This element is independent of the choice of symplectic basis. (It is the image in $\wedge^2 V$ of B under the identification of $\wedge^2 V^*$ with $\wedge^2 V$ induced by B.)

Let $E(\omega) : \wedge V \to \wedge V$ denote the operation of exterior multiplication by ω. So

$$E(\omega) = \sum e_{u_i} e_{v_i}.$$

Let

$$F(\omega) := E(\omega)^\dagger$$

so

$$F(\omega) = \sum e^\dagger_{v_i} e^\dagger_{u_i}.$$

For $\alpha \in \wedge^p V$ we have, by (3),

$$e^\dagger_v e^\dagger_u \alpha = (-1)^{p-1}(-1)^{p-2} \star v e_v e_u \star \alpha = - \star e_v e_u \star \alpha = \star e_u e_v \star \alpha$$

so

$$F(\omega) = \star E(\omega) \star . \tag{14}$$

Alternatively, if $\mu^1, \ldots, \mu^m, \nu^1, \ldots, \nu^m$ is the basis of V^* dual to $u_1, \ldots, v_m$ then

$$F(\omega) = \sum i_{\nu_j} i_{\mu_j}. \tag{15}$$

We now prove the Kaehler-Weil identity

$$[E(\omega), F(\omega)]\alpha = (p-m)\alpha, \quad \alpha \in \wedge^p V. \tag{16}$$

Write

$$E(\omega) = E_1 + \cdots + E_m, \quad E_j := e_{u_j} e_{v_j}$$

and

$$F(\omega) = F_1 + \cdots F_m, \quad F_j = i_{\nu_j} i_{\mu_j}.$$

Let V_j be the two dimensional space spanned by u_j, v_j and write

$$\alpha = \alpha_1 \wedge \cdots \wedge \alpha_p, \quad \alpha_j \in \wedge V_j.$$

Then E_i really only affects the i-th factor since we are multiplying by an even element:

$$E_i(\alpha) = \alpha_1 \wedge \cdots \wedge E_i \alpha_i \wedge \cdots \wedge \alpha_p$$

and F_i annihilates all but the i-th factor:

$$F_i(\alpha) = \alpha_1 \wedge \cdots \wedge F_i \alpha_i \wedge \cdots \wedge \alpha_p.$$

So if $i < j$

$$E_i F_j(\alpha) = F_j E_i(\alpha) = \alpha_1 \wedge \cdots \wedge E_i \alpha_i \wedge \cdots \wedge F_j \alpha_j \wedge \cdots \wedge \alpha_p.$$

In other words,

$$[E_i, F_j] = 0, \quad i \neq j.$$

So

$$[E(\omega), F(\omega)]\alpha = \sum \alpha_1 \wedge \cdots \wedge [E_i, F_i]\alpha_i \wedge \cdots \wedge \alpha_p$$

Since the sum of the degrees of the α_i add up to p, it is sufficient to prove (16) for the case of a two dimensional symplectic vector space with symplectic basis u, v . We need consider three cases, according the possible values of $p = 0, 1, 2$. Let us write E for $F(\omega)$. When $p = 2$, if we apply E to $u \wedge v$ we get 0. So

$$[E, F](u \wedge v) = (EF)(u \wedge v) = E1 = u \wedge v = (2-1) u \wedge v.$$

For $p = 0$ we have $F1 = 0$ so

$$[E, F]1 = -FE1 = -F[u \wedge v] = -1 = (0 - 1) \cdot 1.$$

For $p = 1$ we have $Eu = Ev = Fu = Fv = 0$ so

$$[E, F] = 0 = (1 - 1)\mathrm{id} \quad \text{on} \quad \wedge^1 V. \quad \square$$

Let H act on $\wedge V$ by

$$H = (p - m)\mathrm{id} \quad \text{on } x \wedge^p V. \tag{17}$$

Then, we can write (16) as

$$[E(\omega), F(\omega)] = H. \tag{18}$$

Notice that since $E(\omega)$ raises degree by two,

$$HE(\omega) = (p + 2 - m)E(\omega), \quad E(\omega)H = (p - m)E(\omega)$$

on $\wedge^p V$ so

$$[H, E(\omega)] = 2E(\omega)$$

and similarly

$$[H, F(\omega)] = -2F(\omega).$$

So we can summarize are computations by saying that the assignments

$$F \mapsto F(\omega), \quad H \mapsto H, \quad E \mapsto E(\omega)$$

give a representation of g on $\wedge V$.

¿From now on we shall drop the ω and simply write E and F.

We can enlarge our graded $sl(2)$ to a graded superalgebra as follows: Consider the space $V \otimes \mathbf{R}^2$ (or the space $V \otimes \mathbf{C}^2$ if we are over the complex numbers).The space $\mathbf{R}^2$ (or $\mathbf{C}^2$) has has a symplectic structure which is invariant under $sl(2)$. Since V has a symplectic structure, the tensor product, as the tensor product of two symplectic vector spaces, has an orthogonal structure. Call the corresponding symmetric form, S. Thus, if we choose

$$e := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad f := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

as a symplectic basis of $\mathbf{R}^2$, then

$$S(u \otimes e, v \otimes e) = 0 = S(u \otimes f, v \otimes f), \quad S(u \otimes e, v \otimes f) = B(u \otimes v) = S(v \otimes f, u \otimes e),$$

where $u, v \in V$. Then we can form the superalgebra whose even part is $\mathbf{R}$ (commuting with everything) and whose odd part is $V \otimes \mathbf{R}^2$ with brackets

$$[w, w'] = -S(w, w'), \quad w, w' \in V \otimes \mathbf{R}^2.$$

(This Lie algebra is a super analogue of the Heisenberg algebra.) In particular,

$$u \otimes e, v \otimes e] = 0 = [u \otimes f, v \otimes f], \qquad [u \otimes e, v \otimes f] = -B(u, v).$$

This Lie superalgebra is clearly invariant under the action of the orthogonal group of $V \otimes \mathbf{R}^2$. Put another way, this orthogonal group acts as automorphisms of the superalgebra structure. In particular, $sl(2)$ acts as infinitesimal automorphisms(derivations) of this algebra, and so we can take the semi-direct product of $sl(2)$ with this Lie superalgebra. If we define

$$h_1 := V \otimes e, \quad h_1 := V \otimes f$$

then we obtain a Lie superalgebra

$$g_2 \oplus h_{-1} \oplus (\mathbf{R}H \oplus \mathbf{R}) \oplus h_1 \oplus g_2. \tag{19}$$

Then the map

$$\begin{array}{rcl} u \otimes e & \mapsto & e_u \\ u \otimes f & \mapsto & e_U^\dagger \\ r \in \mathbf{R} & \mapsto & \text{mutliplication by } r \end{array}$$

extends the action of our graded $sl(2)$ to a representation of the Lie superalgebra (19) on $\wedge V$, as can be directly checked. In particular, we have the identity

$$[e_v^\dagger, E] = e_v.$$

6 Hermitian vector spaces.

Let V be a $2m$ dimensional real vector space equipped with a positive definite symmetric bilinear form, B_s and an alternating form B_a which are related by

$$B_a(v, w) = B_s(v, Jw) \tag{20}$$

where

$$J : V \to V$$

satisfies

$$J^2 = -I. \tag{21}$$

The fact that B_a is alternating and B_s is symmetric implies that

$$B_s(v, Jw) + B_s(Jv, w) = 0 \tag{22}$$

which says that J infinitesimally preserves B_s. Replacing w by Jw in this equation gives

$$B_s(Jv, Jw) = -B_s(v, J^2 w) = B_s(v, w)$$

so J preserves B_s, i.e. belongs to the orthogonal group associated with B_s. Also

$$B_a(Jv, Jw) = B_s(Jv, J^2 w) = -B_s(Jv, w) = B_s(v, Jw) = B_a(Jv, Jw)$$

so J belongs to the symplectic group associated to B_a.

Decompose

$$V = V_1 \oplus \cdots \oplus V_m$$

into two dimensional subspaces invariant under J and mutually perpendicular under B_s. For each $i = 1, \ldots, m$ pick a vector $e_i \in V_i$ which satisfies $B_s(e_i, e_i) = 1$, i.e. is a unit vector for the orthogonal form. Let $f_i := -Je_i$. Then

$$B_s(e_i, f_i) = 0, \quad B_s(f_i, f_i) = 1$$

and

$$B_a(e_i, f_i) = -B_s(e_i, J^2 e_i) = B_s(e_i, e_i) = 1$$

while

$$B_a(e_i, e_j) = B_a(e_i, f_j) = B_a(f_i, f_j) = 0, \qquad i \neq j$$

so $e_1, \ldots, e_m, f_1 \ldots, f_m$ is a symplectic basis (for B_a) and an orthonormal basis (for B_s). We take

$$\Omega := e_1 \wedge f_1 \wedge c \cdots \wedge e_m \wedge f_m$$

as our basis of $\wedge^{2m}(V)$ as is our symplectic prescription, and use this to fix the orientation of V as far as the orthogonal form B_s is concerned. We now have two star operators, $\star_a$ corresponding to the symplectic form, B_a and $\star_s$ corresponding to the orthogonal form, B_s. Since J preserves B_s and B_a, and since B_a is related to B_s by (20) we conclude that

$$J\star_s = \star_s J \tag{23}$$

$$J\star_a = \star_a J \tag{24}$$

$$\star_a = \star_s \circ J \tag{25}$$

hold, where we have extended J as usual to act on $\wedge V$. This extended J preserves the (extended) form B_s, i.e.

$$JJ^\dagger = I.$$

On the other hand, $J^2 = (-1)^p$ on $\wedge^p V$ so

$$J^\dagger = J^{-1} = (-1)^p J \quad \text{ on } \quad \wedge^p V. \tag{26}$$

In this formula, $J^\dagger$can mean either the transpose of J with respect to B_s or with respect to B_a since J is orthogonal with respect to B_s and symplectic with respect to B_a.

Direct verification shows that $J\omega = \omega$ where

$$\omega = e_1 \wedge f_1 + \cdots + e_m \wedge f_m$$

is the element of $\wedge^2 V$ corresponding to B_a and hence that

$$[J, E] = 0 \tag{27}$$

where E acts by multiplication by ω. Recall that F acts as the transpose of E with respect to B_a. Taking the transpose with respect to B_a of (27) gives

$$[F, J^{-1}] = 0.$$

Multiplying on the right and left by J gives

$$[J, F] = 0. \tag{28}$$

Since E and F generate g, we conclude from (27) and (28) that J commutes with the entire $sl(2)$ action.

According to (14)

$$F = \star_a E \star_a = \star_a^{-1} E \star_a$$

since $\star_a^2 = I$. From (28) and (25) we conclude that

$$F = \star_s^{-1} E \star_s\ . \tag{29}$$

Since J lies in the Lie algebra of the orthogonal group of B_s, the one parameter group $t \mapsto \exp tJ$ is a one parameter group of orthogonal transformations of V and so extends to a one parameter group of one parameter group of orthogonal transformations of $\wedge V$ which commute with $\star_s$:

$$(\exp tJ)\star_s = \star_s(\exp tJ).$$

Differentiating this equation with respect to t and setting $t = 0$ gives

$$J^\sharp \star_s = \star_s J^\sharp \tag{30}$$

where $J^\sharp$ is the derivation induced by J on the exterior algebra, i.e.

$$J^\sharp(v_1 \wedge \cdots \wedge v_p) = Jv_1 \wedge v_2 \wedge \cdots \wedge v_p + v_1 \wedge Jv_2 \wedge \cdots \wedge v_p + \cdots + v_1 \wedge \cdots \wedge Jv_p.$$

Let $V_{\mathbf{C}} := V \otimes \mathbf{C}$ denote the complexification of V and extend all maps from V to $V_{\mathbf{C}}$ or from $\wedge V$ to $\wedge V_{\mathbf{C}}$ so as to be complex linear. For example, J has eigenvalues i and $-i$ on $V_{\mathbf{C}}$ and we can write

$$V_{\mathbf{C}} = V^{1,0} \oplus V^{0,1}$$

where $V^{1,0}$ consist of all vectors of the form $v - iJv,\ v \in V$ and is the eigenspace with eigenvalue i for J and $V^{0,1}$ consist of all vectors of the form $v + iJv,\ \ v \in V$ and is the eigenspace with eigenvalue $-i$. Both of these are complex subspaces of $V_{\mathbf{C}}$ and hence we have the complex decomposition of the complex exterior algebra

$$\wedge V_{\mathbf{C}} = \bigoplus \wedge^{p,q}, \qquad \wedge^{p,q} := \wedge^p(V^{1,0}) \otimes \wedge^q(V^{0,1}).$$

For example,

$$J = i^{p-q} I \quad \text{on} \quad \wedge^{p,q}$$

so that since $J\omega = \omega$ and $\omega \in \wedge^2 V_{\mathbf{C}}$ we conclude that

$$\omega \in \wedge^{1,1}.$$

Therefore

$$E : \wedge^{p,q} \to \wedge^{p+1,q+1}. \tag{31}$$

Similarly,

$$J^\sharp = (p-q)iI \quad \text{on} \quad \wedge^{p,q}$$

Since $\star_s : \wedge^k (V_{\mathbf{C}}) \to \wedge^{2m-k}(V_{\mathbf{C}}$ and (30) holds we conclude that

$$\star_s : \wedge^{p,q} \to \wedge^{m-q,m-p}. \tag{32}$$

Finally, it follows from (31), (29), and (32) that

$$F : \wedge^{p,q} \to \wedge^{p-1,q-1}. \tag{33}$$

7 Symplectic Hodge theory.

Let (X, ω) be a $2m$-dimensional symplectic manifold. For $\alpha, \beta \in \Omega(X)^p_0$ define

$$\langle \alpha, \beta \rangle := \int_X \alpha \wedge \star\beta.$$

For $\gamma \in \Omega^{p-1}(X)_0$ we have

$$\begin{array}{rcl} d(\gamma \wedge \star\beta) & = & d\gamma \wedge \star\beta + (-1)^{p-1} \gamma \wedge d \star \beta \\ & = & d\gamma \wedge \star\beta + (-1)^{p-1} \gamma \wedge \star(\star d \star)\beta \quad \text{so} \\ \langle d\gamma, \beta \rangle & = & \langle \gamma, d^\dagger \beta \rangle \end{array}$$

with, $d^\dagger$, the transpose of d with respect to $\langle\ ,\ \rangle$ given by

$$d^\dagger = (-1)^p \star d \star.$$

We define

$$\delta := d^\dagger = (-1)^p \star d \star. \tag{34}$$

The symbol of the first order differential operator , d, is given by

$$\sigma(d)(\xi) = e_\xi, \quad \xi \in T^*(X)_x.$$

Hence the symbol of δ is given by

$$\sigma(\delta)(\xi) = e_\xi^\dagger.$$

Let E act on $\Omega(X)_0$ pointwise as $E(\omega)$, that is as the operator consisting of exterior multiplication by ω. We claim that

$$[\delta, E] = d. \tag{35}$$

Proof. Since δ is a first order differential operator, and E is a zeroth order differential operator, the symbol of $[\delta, E]$ is given by

$$\sigma\left([\delta, E]\right)(\xi) = [\sigma(\delta)(\xi), E] = [e_\xi^\dagger, E] = e_\xi = \sigma(d)(\xi).$$

Thus

$$d - [\delta, E]$$

is a zeroth order differential operator. So to show that it vanishes, it is enough to find local coordinates, $w^1, \ldots, w^{2m}$ about each point such that this zeroth order differential operator annihilates all the dw^I. Now the operator d annihilates the dw^I in any coordinate system. By Darboux's theorem, we may choose local coordinates such that

$$\omega = dw^1 \wedge dw^{m+1} + \cdots + dw^m \wedge dw^{2m}.$$

In these coordinates, the operator $\star$ has constant coefficients when applied to any of the dw^I, and hence it follows from (34) that $\delta dw^I = 0$ as well. Thus both sides of (35) vanish when applied to dw^I, completing the proof of (35). $\square$

We let F act as $E^\dagger$. Taking the transpose of (35) we get

$$\delta = [d, F]. \tag{36}$$

Next we prove that

$$\delta^\dagger = -d. \tag{37}$$

Proof. Let $\alpha \in \Omega(X)_0^{p-1}, \quad \beta \in \Omega^p(X)_0$. Then

$$\begin{aligned}
\langle \delta\beta, \alpha \rangle &= (-1)^{p-1}\langle \alpha, \delta\beta \rangle \\
&= (-1)^{p-1}\langle d\alpha, \beta \rangle \\
&= (-1)^{p-1}(-1)^p\langle \beta, d\alpha \rangle \\
&= \langle \beta, -d\alpha \rangle. \quad \square.
\end{aligned}$$

Thus

$$\begin{aligned}
(E\delta)^\dagger &= -dF && (38)\\
(\delta E)^\dagger &= -Fd && (39)\\
\delta d = \delta[\delta, E] &= \delta(\delta E - D\delta) = -\delta E \delta \\
d\delta = [\delta, E]\delta &= \delta E \delta \quad \text{so} \\
d\delta + \delta d &= 0. && (40)
\end{aligned}$$

We can view the last of these equations as saying that the symplectic analogue of the (Hodge) Laplacian vanishes.

We can summarize all the results in this section by introducing a large superalgebra: Let $\mathcal{V} = \mathcal{V}(X)$ denote the space of all vector fields on X, and let $\Omega^1 = \Omega^(X)$ denote the space of one forms. The symplectic form induces an isomorphism

$$\mathcal{V} \to \Omega^1, \quad \xi \mapsto \gamma$$

such that

$$i_\xi^\dagger = e_\gamma.$$

Let $\mathcal{F} = \mathcal{F}(X) = \Omega^0(X)$ denote the space of smooth functions on X. Then we get a Lie superalgebra $\hat{g}$ acting on $\Omega(M)$ where

$$\begin{aligned} \hat{g}_{-2} &:= \mathbf{R}F \\ \hat{g}_{-1} &:= \mathcal{V} \oplus \mathbf{R}\delta \\ \hat{g}_0 &:= \mathcal{V} \oplus \mathbf{R}H \oplus \mathcal{F} \\ \hat{g}_1 &:= \Omega^1 \oplus \mathbf{R}d \\ \hat{g}_2 &:= \mathbf{R}E. \end{aligned}$$

We list several of the bracket relations, the others have already been given, or can be obtained by taking the transpose: The element of $\mathcal{V} \subset \hat{g}_0$ corresponding to the element $\xi \in \mathcal{V}$ will be denoted by L_ξ and acts by Lie derivative. The element of $\hat{g}_{-1}$ corresponding to ξ is denoted by i_ξ and acts by interior product. We have the bracket relation

$$[i_\xi, d] = L_\xi$$

which is just the Weil identity. The element of $\Omega^1 \subset \hat{g}_1$ corresponding to $\gamma \in \Omega^1$ is denoted by e_γ. As already mention, it acts pointwise as exterior multiplication by γ. We have

$$[i_\xi, e_\gamma] = \gamma(\xi) \in \mathcal{F} \subset \hat{g}_0.$$

It acts by pointwise multiplication.

8 Excursus on $sl(2)$ modules.

Will will need some facts about $sl(2)$ modules for our study of the Lefschetz theorem in the next section. The action of $sl(2)$ on $\Omega(X)$ described in the preceding section is such that H acts as multiplication by $p-m$ on $\Omega^p(X)$. Thus although $\Omega(X)$ is an infinite dimensional vector space, it is a finite direct sum of (infinite dimensional) vector spaces on each of which H acts as a scalar. We axiomatize this property, recalling that g denotes the graded $sl(2)$, in particular it denotes $sl(2)$ with a specific choice of H:

Definition. *A g module A is of* **finite H type** *if V is a finite direct sum of vector spaces,*

$$V = V_1 \oplus \cdots \oplus V_k$$

such that H acts as scalar multiplication by λ_i on V_i and

$$\lambda_i \neq \lambda_j, \quad i \neq j.$$

The projection $\pi_i : V \to V_i$ corresponding to this decomposition is given by

$$\frac{1}{\prod_{i \neq njj}(\lambda_i - \lambda_j)} \prod (H - \lambda_i).$$

Therefore, π_i carries every g submodule into itself. In particular, any submodule and any quotient module of a g-module of finite H type is again of finite H type.

If an element v in any H module satisfies $Hv = \lambda v$, then the bracket relations imply that

$$HEv = (\lambda + 2)Ev, \quad \text{and} \quad HFv = (\lambda - 2)Fv.$$

Since $[E, F] = H$, it follows that $[E, F]v = \lambda v$ and then by induction on k that

$$[E, F^k]v = k(\lambda - k + 1)F^{k-1}v. \tag{41}$$

Indeed, for $k = 1$ this is just the assertion $[E, F]v = \lambda v$, and assuming the result for k, we have

$$\begin{aligned}
[E, F^{k+1}]v &= EF^{k+1}v - F^{k+1}Ev \\
&= (EF)F^k v - (FE)F^k v + F[E, F^k]v \\
&= HF^k v + k(\lambda - k + 1)F^k v \\
&= (\lambda - 2k + k(\lambda - k + 1))\, Y^k v \\
&= (k + 1)(\lambda - k)F^k v. \qquad \square
\end{aligned}$$

¿From this we can conclude that

Every cyclic g-module of finite H type is finite dimensional.

Proof. Let v generate V as a $U(g)$ module. Decompose v into its components of various types:

$$v = v_1 + \cdots + v_k, \quad v_i \in V_i.$$

It is enough to show that the submodule generated each v_r is finite dimensional. By Poincaré-Birkhhoff-Witt, this module is spanned by the vectors $F^i E^j H^k v_i$. Since Hv is a multiple of v, it is enough to consider $F^i E^j v_r$. Now $HE^j v_r = (\lambda_r + 2j)E^j v_r$. Since there are only finitely many possible eigenvalues of H (by the definition of finite H type) it follows that $E^j v = 0$ for $j >> 0$. If j is such that $E^j v \neq 0$, then $H(F^i E^j v_r) = (\lambda_r + 2j - 2i)F^i E^j v_r$, so we conclude that for each such j there are only finitely many i with $F^i E^j v_r \neq 0$. In short, there are only finitely many non-zero $F^i E^j v_r$, proving that the submodule generated by v_r is finite dimensional.

If we don't want to use the Poincaré-Birkhhoff-Witt theorem, we can proceed as follows: We have shown that there are only finitely many non-zero $F^i E^j v_r$. We must show that they span the submodule generated by v_r. Applying F gives $F^{i+1} E^j v_r$ which is of the same type. Applying H carries each such term into a multiple of itself. So we need only check what happens when we apply E. We have

$$EF^i E^j = F^i E^{j+1} v_r + i(\lambda_r + 2j - i + 1)F^{i-1} E^j v_r$$

by (41). $\square$

As immediate consequences of this result we can deduce that

- Every irreducible g-module of finite H type is finite dimensional.
- Every cyclic g-module of finite H type is a finite direct sum of irreducibles.

(The second statement is true for any finite dimensional g-module.)

Suppose the λ_i are real and we have labeled them in decreasing order. Then, if $v \in V_1$ we must have $Ev = 0$ and, by (41),

$$EF^r v = f(\lambda_1 - r + 1)F^{r-1}v.$$

This shows that the vectors $F^r v$ span a submodule of V. Now suppose that V is irreducible. Then the submodule spanned by these vectors is all of V. We have

$$HF^r v = (\lambda - 2r)F^r v$$

and, since V is of finite H type, we must have $F^\ell v = 0$ for some ℓ. Let ℓ_0 be the smallest such ℓ, so that $F^\ell v = 0, \quad \forall \ell \geq \ell_0,$, but $F^{\ell_0 - 1}v \neq 0$. Set $j := \ell_0 - 1$ and

$$v_i := F^i v, \quad i = 0, \dots, j.$$

These vectors are linearly independent since they correspond to different eigenvalues of H, and they span all of V; i.e. they are a basis of V. Also,

$$Fv_j = F^{\ell_0} v = 0.$$

Applying E to this equation and using (41) we conclude that

$$(j+1)(\lambda_1 - j)v_j = 0,$$

implying that

$$\lambda_1 = j.$$

In terms of this basis we have,

$$\begin{array}{rcl} Hv_i & = & (j - 2i)v_i \\ Fv_i & = & v_{i+1} \\ Ev_i & = & i(j - i + 1)v_{i-1} \end{array}$$

for $i = 0, \dots, j$. These equations completely determine the representation. Conversely every finite dimensional representation of g is of this form, as can easily be verified from the above equations. We have just repeated some well known facts about the irreducible finite dimensional representations of $sl(2)$.

We now return to the consideration of a (possibly infinite dimensional) g-module V of finite H type

$$V = V_1 \oplus \cdots \oplus V_k.$$

Let us call an element *homogeneous* if it belongs to one of the summands in this decomposition. Let us call an element $v \in V$ **primitive** if it is homogeneous and satisfies

$$Ev = 0.$$

Repeating the same proof given above (which only used the finite H type property) we see that eventually $F^\ell v = 0$ if v is primitive and that the cyclic module generated by v is finite dimensional and that

$$Hv = kv$$

where $k + 1$ is the dimension of the cyclic submodule of V generated by the primitive element v.

We can now state and prove some important structural properties of a g module V of finite H type:

1. Every $v \in V$ can be written as a finite sum
$$v = \sum F^r v_r, \qquad v_r \text{ primitive.} \tag{42}$$

2. The eigenvalues of H are all integers. Hence by relabeling, we may decompose
$$V = \bigoplus V_r, \quad H = r\mathrm{Id} \text{ on } V_r.$$
We may then write
$$V = V_{even} \oplus V_{odd}$$
where
$$V_{even} := \bigoplus_{r\ even} V_r, \qquad V_{odd} := \bigoplus_{r\ odd} V_r.$$

3. The map
$$F^k : V_k \to V_{-k}$$
is bijective.

4. an element $v \in V_r$, $r \geq 0$ is primitive if and only if
$$F^{r+1} v = 0.$$

Proofs.

1. We may replace V by the cyclic module generated by v in proving 1. This is a submodule of V and hence of finite H type. Being also cyclic, it is finite dimensional. We may therefore decompose it into a finite sum of irreducibles, and write v as a sum of its components in these irreducibles. But each element of an irreducible is a sum of the desired form as proved above. Hence v is.

2. The decomposition in 1. and its proof show that the only possible eigenvalues for H are integers, since this is true for finite dimensional irreducible representations.

3. We know that this is true for irreducibles, hence for any direct sum of irreducibles, hence for any cyclic module of finite H type. Now consider the general case: If $v \in V_{-k}$, consider the cyclic module generated by v. The bijectivity property for this submodule implies that there is some $w \in V_k$ such that $F^k w = v$. This shows that the map $F^K : V_k \to V_{-k}$ is surjective. Similarly, to prove that this map is injective, consider the cyclic submodule generated by $v \in V_k$. If $F^k v = 0$ we conclude that $v = 0$. Hence the map is injective as well.

4. If v is primitive, the submodule it generates is finite dimensional of dimension $r + 1$ as we have seen above. Hence the necessity follows from the structure of finite dimensional irreducibles. To prove the sufficiency, decompose v as in 1. Let u be the term corresponding to $\ell = 0$ in this decomposition, so $u \in V_r$ is primitive, and this decomposition implies that

$$v = u + Fw, \qquad w \in V_{r+2}.$$

Since $u \in V_r$ is primitive, we know that $F^{r+1}u = 0$. Hence

$$0 = F^{r+1}v = F^{r+2}w.$$

Since $w \in V_{r+2}$ and $F^{r+2} : V_{r+2} \to F_{-r-2}$ is bijective, we conclude that $w = 0$. Hence $v = u$ is primitive. $\square$

We will also want to use items 2) and 3) with the roles of E and F interchanged (which can be arranged by an automorphism of $sl(2)$ so that

$$E^k : V_{m-k} \to V_{m+k} \qquad \text{is bijective} \tag{43}$$

and

$$\text{If } v \in V_{m-k} \ \text{ then } E^{k+1}v = 0 \ \Leftrightarrow \ Fv = 0. \tag{44}$$

9 The strong Lefschetz property.

We return the study of a $2m$ dimensional symplectic manifold, X and the action of g on $\Omega = \Omega(X)$. Since $[E, d] = 0$, E carries closed forms into closed forms and exact forms into exact forms, and hence induces a map on cohomology which we shall denote by $[E]$. so

$$[E] : H^p(X) \to H^{p+2}(X).$$

In particular,

$$[E]^k : H^{m-k}(X) \to H^{m+k}(X).$$

We say that X has the **strong Lefschetz property** if this map is surjective for all k.

A form α is called **harmonic** if

$$d\alpha = 0 = \delta\alpha.$$

We shall denote the space of all harmonic forms by Ω_{har}. Suppose that α is harmonic. Since $[d, E] = 0$, we conclude that $dE\alpha = 0$. Since $[d, F] = \delta$, and $\delta\alpha = 0$, we conclude that $dF\alpha = 0$. A similar argument from the bracket relations $[\delta, F] = 0$, $[\delta, E] = d$ shows that $\delta E\alpha = 0 = \delta F\alpha$. In short,

$$\Omega_{har} \quad \text{is a} \quad g \text{ submodule of } \quad \Omega.$$

In particular, it is of finite H type and hence

$$E^k : \Omega_{har}^{m-k} \to \Omega_{har}^{m+k} \quad \text{is bijective} \tag{45}$$

for all k. Furthermore, for $\mu \in \Omega^{m-k}$,

$$E^{k+1}\mu = 0 \Leftrightarrow F\mu = 0. \tag{46}$$

A symplectic manifold X is said to satisfy the **Brylinski condition** if every cohomology class has a representative which is harmonic.

Theorem (Mathieu). *A symplectic manifold satisfies the Brylinski condition if an only if it has the strong Lefschetz property.*

Proof (Dang Yan).

Brylinski $\Rightarrow$ Lefschetz. Consider the commutative diagram

$$\square[\Omega_{har}^{m-k}`\Omega_{har}^{m+k}`H^{m-k}(X)`H^{m+k}(X); E^k```[E]^k].$$

The Brylinski condition says that the vertical arrows are surjective, and (45) says that the top line is bijective. Hence the bottom row is surjective.

Lefschetz $\Rightarrow$ Brylinksi. Let $c \in H^{m-k}(X)$. Consider $[E]^{k+1}c \in H^{m+k+2}(X)$. By the strong Lefschetz condition, we can write $[E]^{k+1}c = [E]^{k+2}c_2$ where $c_2 \in H^{m-k-1}(X)$. We can therefore write any element of $H^{m-k}(X)$ as

$$c = c_1 + [E]c_2, \quad c_2 \in H^{m-k-1}(X), \quad [E]c_1 = 0, \tag{47}$$

where we take $c_1 = c - [E]c_2$.

Next observe that it is enough to prove that cohmology classes in degree $\leq m$ have harmonic representatives. Indeed, if $c \in H^{m+k}(X)$ then $c = [E]c'$, $c' \in H^{m-k}(X)$ and a harmonic representative for c' is carried by E^k into a harmonic representative for c. If $c \in H^0(X)$ or $H^1(X)$, then $[F]c = 0$ since $[F]$ lowers degree by two. If μ is a closed form representing c, so that $d\mu = 0$, the $\delta\mu = [d, F]\mu = 0$ and μ is harmonic. So we need only prove the Brylinski propoerty for cohomology classes of degree $2 \leq p \leq m$. We will proceed by induction on

p usin (47). By induction, c_2 has a harmonic representative, call it θ, so that $E\theta$ is a harmonic representative of $[E]c_2$. So we need only prove that c_1 has a harmonic representative.

In other words, dropping the subscript, we need only prove that any $c \in H^p(X), p = m - k$ satisfying $[E]^{k+1}c = 0$ has a harmonic representative. Let $\mu \in \Omega^{m-k}(X)$ be a representative of c. So $d\mu = 0$ and $E^{k+1}\mu = d\beta,\ \beta \in \Omega^{m+k+1}(X)$. Since

$$E^{k+1} : \Omega^{m-k-1}(X) \to \Omega^{m+k+1}(X)$$

is bijective, $\beta = E^{k+1}\alpha$ where $\alpha \in \Omega^{m-k-1}(X)$ and

$$E^{k+1}\mu = d\beta = E^{k+1}d\alpha.$$

Replkace μ by $\nu := \mu - d\alpha$. Then ν is again a reperesentative of c, and

$$E^{k+1}\nu = E^{k+1}\mu - E^{k+1}d\alpha = 0$$

so

$$F\nu = 0$$

by (44). But then

$$\delta\nu = [d, F]\nu = 0$$

so ν is a harmonic representative for c. $\square$

Remarks.

1. If X is compact, Poincaré duality implies that dim $H^{m-k}(X)$ = dim $H^{m+k}(X)$. So the strong Lefschetz condition asserts that

   $$[E]^k : H^{m-k}(X) \to H^{m+k}(X)$$

   is bijective.

2. In particular, if X is compact and satisfies the strong Lefschetz condition, we may define a bilinear pairing on $H^{m-k}(X)$ by mapping the pair (c_1, c_2) into $H^{2m}(X)$ by

   $$(c_1, c_2) \mapsto [E]^k(c_1 \cdot c_2) = \big([E]^k c_1\big) \cdot c_2$$

   (recall that E^k is just multiplication by ω^k). We may identify $H^{2m}(H)$ with $\mathbf{R}$ (or $\mathbf{C}$) using the symplectic volume form,. Composing the this identification with the above bilinear map, we get a bilinear form, call it K. We claim that K is non-degenerate. Indeed, if $E^k c_1 \cdot c_2 = 0$ for all c_2, then, by Poincaré duality, $E^k c_1 = 0$ and hence, by Strong Lefschetz, $c_1 = 0$.

3. By construction, the bilinear form K is alternating when $m-k$ is odd and symmetric when $m-k$ is even. For an alternating form to be nondegenerate, the underlying vector space must be even dimensional. So, if X is compact and satisfies the strong Lefschetz condition, all its odd degree Betti numbers are even.

4. We will see below that a Kaehler manifold always satisfies the Brylinski condition, by showing that a form is harmonic with respect to the symplectic piece of the Kaehler form if and only if it harmonic in the Riemannian sense, and using the fact that in a Riemannian manifold, every cohomology class has a harmonic representative. Thus Kaheler manifolds always satisfy the strong Lefschetz condition.

5. Many years ago Thurston produced an example of a sympletic four manifold whose first Betti number is odd. This shows that the Brylinski condition does not hold for all symplectic manifolds.

6. On a symplectic manifold we can replace the de Rham cohomology by defining $H^p(X)_{symp} \subset H^p(X)$ to consist of those classes which *are* the images of harmonic forms. It follows from the preceding discussion that the strong Lefschetz condition holds when we replace H^p by H^p_{symp}.

10 Riemannian Hodge theory.

Let X be a compact oriented Riemann manifold of dimension n. We denote the metric by B or (in the next section) by B_s. The $\star$ operator acting pointwise on $\wedge T^*(X)$ gives an operator, also denoted by $\star$ (or by $\star_s$ in the next section)

$$: \star : \Omega^p(X) \to \Omega^{n-p}(X)$$

satisfying

$$\star^2 = (-1)^{p(n-p)} I. \quad \text{on} \Omega^p.$$

There is an l^2 inner product on forms given by

$$\langle \alpha, \beta \rangle = \int_X (B(\alpha, \beta)_x dx = \int_X \alpha \wedge \star\beta, \quad \alpha, \beta \in \Omega^p(X),$$

where dx denotes the volume form (and where forms of differing degrees are orthogonal).

If $\alpha \in \Omega^{p-1}, \beta \in \omega^p$ then

$$d(\alpha \wedge \star\beta = d\alpha \wedge \beta + (-1)^{p-1}\alpha \wedge \star(\star^{-1} d\star)\beta.$$

Integrating this over X and using Stokes gives

$$\langle d\alpha, \beta \rangle = \langle \alpha, d^\dagger \beta \rangle$$

where

$$d^\dagger = (-1)^p \star^{-1} d \star .$$

We define

$$\delta := d^\dagger, \quad \Delta := d\delta + \delta d$$

and observe that Δ is self adjoint. The symbol of d at $\xi \in T^*X_x$ is given by e_ξ and the symbol of δ is given by $e_\xi^\dagger$ so the symbol of Δ at ξ is

$$\sigma(\Delta)(\xi) = e_\xi e_\xi^\dagger + e_\xi^\dagger e_\xi = B(\xi, \xi)_x I,$$

so Δ is elliptic. We may apply the theory of elliptic operators to conclude that

- The kernel of Δ is finite dimensional.
- There exists a Greens operator $G : \Omega^p \to \Omega^p$ which is self adjoint and whose image is orthogonal to ker Δ and a projection $H : \Omega^p \to \ker \Delta$ and such that
$$u = \Delta G u + Hu, \quad \forall u \in \Omega^p.$$
- This gives the **Hodge decomposition** of $u \in \Omega^p$ into three mututally orthogonal pieces:
$$u = u_1 + u_2 + u_3, \quad \forall u \in \Omega^p$$
where
$$u_1 := Hu \quad \text{is harmonic}$$
i.e. lies in ker Δ,
$$u_2 := d\delta G u$$
is exact, and
$$u_3 := \delta d G u$$
is coexact.
- In particular, since the image of δ is orthogonal to the closed forms, we see that $u_3 = 0$ is u is closed, and hence every cohomology class has a unique harmonic representative.

11 Kaehler Hodge theory.

Let X be a compact kaehler manifold of dimension $n = 2m$. This means that we are given three pieces of data: 1) a Riemann metric, which we may conisider as providing a poisitive definite symmetric form, $B_s(\ ,\ )_x$ on each cotangent space, T^*X_x, an antisymmetric form $B_a(\ ,\ )_x$ and

$$J : T * X \to T^X$$

which is a bundle map satsifying

$$J^2 = -I.$$

These pieces are related (at each cotangent space) as in section 6. In addition there is the Kaehler integrability condition, one consequence (version) of which is

$$d\omega = 0$$

where ω is the two form (section of $\wedge^2 T^*X$) associated with B_a as in section 6. We will return to this doncition of integrability later.

Thus X is both a Riemannian manifold and a symplectic manifold. So it has a star operator, $\star_s$ associated to the Riemann metric, and and a star operator $\star_a$ associated to the symplectic form, both map

$$\Omega^p \to \Omega^{n-p}$$

and are related by

$$\star_a = \star_s \circ J.$$

We have

$$\langle \alpha, \beta \rangle_s := \int_X \alpha \wedge \star_s \beta$$

and

$$\langle \alpha, \beta \rangle_a := \int_X \alpha \wedge \star_a \beta$$

which, in view of the pointwise relation between $\star_a$ and $\star_a$ are related by

$$\langle \alpha, \beta \rangle_a = \langle \alpha, J\beta \rangle_s$$

or, equivalently

$$\langle \alpha, \beta \rangle_s = \langle \alpha, J^{-1}\beta \rangle_a.$$

Let $\delta = \delta_s$ denote the transpose of d with repsect to $\langle \ , \ \rangle_s$ and let δ_a denote the transpose of d with respect to $\langle \ , \ \rangle_a$. They are related by

$$\delta = J\delta_a J^{-1}. \tag{48}$$

Indeed,if $\alpha \in \Omega^{p-1},\ \beta \in \Omega^p$,

$$\begin{aligned} \langle d\alpha, \beta \rangle_s &= \langle d\alpha, J^{-1}\beta \rangle_a \\ &= \langle \alpha, \delta_a J^{-1}\beta \rangle_a \\ &= \langle \alpha, J^{-1}(J\delta_a J^{-1})\beta \rangle_a \\ &= \langle \alpha, (J\delta_a J^{-1})\beta \rangle_s. \quad \square \end{aligned}$$

Now

$$J^{-1} = (-1)^p J \quad \text{on } \Omega^p.$$

So, on Ω^p,

$$\begin{aligned}\delta &= (-1)^p J\delta_a J \\ &= (-1)^p(-1)^{p-1}J^{-1}\delta_a J \\ &= -J^{-1}\delta_a J\end{aligned}$$

so we can also write (48) as

$$\delta = -J^{-1}\delta_a J. \tag{49}$$

Recall that $g = sl(2)$ acts on $\Omega(X)$ with E acting as multiplication by ω and that J commutes with this action. We have

$$d = [\delta_a, E]$$

Conjugating by J^{-1} gives

$$J^{-1}dJ = [J^{-1}\delta_a J, E] = -[\delta, E].$$

Setting

$$d^c := J^{-1}dJ$$

we obtain

$$[\delta, E] = -d^c. \tag{50}$$

We recall from section 6 that $F = E^\dagger$ where the transpose is taken either with respect to B_s or B_a and

$$F = E^\dagger = \star_r^{-1} E \star_r .$$

Since $J^\dagger = J^{-1}$ we have, taking transposes with respect to the Riemann structure, B_s,

$$(d^c)^\dagger = \left(J^{-1}dJ\right)^\dagger = J^{-1}\delta J.$$

So if we define

$$\delta^c := J^{-1}\delta J$$

we have

$$[d, F] = \delta^c. \tag{51}$$

To summarize, we have

$$\begin{aligned}[d, E] &= 0 \\ [d, F] &= \delta^c \\ [\delta, E] &= -d^c \\ [\delta, F] &= 0.\end{aligned}$$

We also recall that we have a decomposition

$$\Omega(X) \otimes \mathbf{C} = \bigoplus \Omega^{p,q}$$

and

$$E : \Omega p, q \to \Omega^{p+1,q+1}, \qquad F : \Omega^{p,q} \to \Omega^{p-1,q-1}.$$

Up until now, we have not made use of integrability. Now let us assume that X is indeed a complex manifold and that in terms of holomorphic local coordinates J is equivalent to the standard complex structure on $\mathbf{C}^m$. Then in terms of such coordinates, $z^1, \dots, z^m$ every $\alpha \in \Omega^{p,q}$ can be written as

$$\alpha = \sum \alpha_{K,L} dz^K \wedge d\overline{z}^L$$

where

$$K = (k_1, \dots, k_p), \quad k_1 < \cdots < k_p$$

and

$$L = (\ell_1, \dots, \ell_q), \quad \ell_1 < \cdots < \ell_q.$$

¿From this we deduce that

$$d\alpha = \partial\alpha + \overline{\partial}\alpha, \quad \partial\alpha \in \Omega^{p+1,q}, \quad \overline{\partial}\alpha \in \Omega^{p,q+1}. \tag{52}$$

This is the key property that we will use. Continuing with the assumption that $\alpha \in \Omega^{p,q}$, we have

$$\begin{aligned} d^c\alpha &:= J^{-1} dJ\alpha \\ &= i^{p-q}\left(J^{-1}\partial\alpha + J^{-1}\overline{\partial}\alpha\right) \\ &= i^{p-q}\left(i^{q-p-1}\partial\alpha + i^{q+1-p}\overline{\partial}\alpha\right) \\ &= (1/i)\left(\partial\alpha - \overline{\partial}\alpha\right). \end{aligned}$$

Thus

$$id^c = \partial - \overline{\partial}. \tag{53}$$

Now $d^2 = 0$ implies that $\partial^2 = \overline{\partial}^2 = 0$. Thus

$$\begin{aligned} idd^c &= (\partial + \overline{\partial})(\partial - \overline{\partial}) \\ &= \overline{\partial}\partial - \partial\overline{\partial} \\ id^c d &= (\partial - \overline{\partial})(\partial + \overline{\partial}) \\ &= \partial\overline{\partial} - \overline{\partial}\partial \end{aligned}$$

so

$$dd^c + d^c d = 0. \tag{54}$$

Also, since $[E, d] = 0$,

$$\begin{aligned} [E, d\delta] &= d[E, \delta] \\ &= dd^c \\ [E, \delta d] &= [E, \delta]d \\ &= d^c d \quad \text{so} \\ [E, d\delta + \delta d] &= 0. \end{aligned}$$

In other words,

$$[E, \Delta] = 0. \tag{55}$$

Now

$$[E, \delta] = [E, d^\dagger] = [E, \partial^\dagger] + [E, \overline{\partial}^\dagger].$$

Since $\partial^\dagger$ is of bidegree $(-1, 0)$ and E is of bidegree $(1, 1)$ the first term on the right of the preceding equation is of bidegree $(0, 1)$ and similarly the second is of bidegree $(1, 0)$. On the other hand, by (50) we have

$$[E, \delta] = d^c = -i\partial + i\overline{\partial}.$$

Comparing the terms of the same bidegree we obtain

$$\begin{aligned} [E, \partial^\dagger] &= i\overline{\partial} && (56) \\ [E, \overline{\partial}^\dagger] &= -i\partial. && (57) \end{aligned}$$

Now $\delta^2 = (\partial^\dagger + \overline{\partial}^\dagger)^2 = 0$ implies that

$$\begin{aligned} \partial^{\dagger 2} &= 0 \\ \overline{\partial}^{\dagger 2} &= 0 \\ \partial^\dagger \overline{\partial}^\dagger + \overline{\partial}^\dagger \partial^\dagger &= 0 \end{aligned}$$

by looking at the terms of differing bidegree. Bracketing the first of these equations with E and using (56) gives

$$0 = [E, \partial^{\dagger 2}] = [E, \partial^\dagger]\partial^\dagger + \partial^\dagger [E, \partial^\dagger]$$

or

$$\overline{\partial}\partial^\dagger + \partial^\dagger \overline{\partial} = 0. \tag{58}$$

Taking complex conjugates gives

$$\partial \overline{\partial}^\dagger + \overline{\partial}^\dagger \partial = 0. \tag{59}$$

Define

$$\begin{aligned} \Delta_\partial &:= \partial\partial^\dagger + \partial^\dagger\partial \\ \Delta_{\overline{\partial}} &:= \overline{\partial}\overline{\partial}^\dagger + \overline{\partial}^\dagger\overline{\partial} \\ &\text{so} \\ \Delta &= d\delta + \delta d \\ &= (\partial + \overline{\partial})(\partial^\dagger + \overline{\partial}^\dagger) + (\partial^\dagger + \overline{\partial}^\dagger)(\partial + \overline{\partial}) \\ &= \Delta_\partial + \Delta_{\overline{\partial}} + \partial\overline{\partial}^\dagger + \overline{\partial}^\dagger\partial + +\partial^\dagger\overline{\partial} + \overline{\partial}\partial^\dagger \\ &= \Delta_\partial + \Delta_{\overline{\partial}}, \end{aligned}$$

In short

$$\Delta = \Delta_\partial + \Delta_{\overline{\partial}}. \tag{60}$$

No let us bracket E with the left side of (59). We have

$$\begin{aligned}
[E, \partial^\dagger \overline{\partial}^\dagger] &= [E, \partial^\dagger]\overline{\partial}^\dagger + \partial^\dagger [E, \overline{\partial}^\dagger] \\
&= i\left(\overline{\partial}\overline{\partial}^\dagger - \partial^\dagger \partial\right) \\
[E, \overline{\partial}^\dagger \partial^\dagger] &= [E, \overline{\partial}^\dagger]\partial^\dagger + \overline{\partial}^\dagger [E, \partial^\dagger] \\
&= i\left(\overline{\partial}^\dagger \overline{\partial} - \partial \partial^\dagger\right) \\
&\qquad \text{so} \\
0 &= i\left(\Delta_{\overline{\partial}} - \Delta_\partial\right) \quad \text{or} \\
\Delta_{\overline{\partial}} &= \Delta_\partial .
\end{aligned}$$

In other words

$$\Delta_\partial = \Delta_{\overline{\partial}} = \frac{1}{2}\Delta. \tag{61}$$

Up to the inessential factor of $\frac{1}{2}$ all three Laplacians are the same. In particular, the harmonic forms are bigraded:

$$H^k_\Delta = \bigoplus_{p+q=k} H^{p,q}_\Delta, \qquad H^{p,q}_\Delta = H^{p,q}_{\Delta_{\overline{\partial}}}. \tag{62}$$

But

$$H^k_\Delta = H^k(X, \mathbf{C})$$

by what we know for Riemannian manifolds, and

$$H^{p,q}_{\Delta_{\overline{\partial}}} = H^{p,q}_{Dol} := H^q(X, \tilde{\Omega}^p),$$

where $\tilde{\Omega}^p$ is the sheaf of holomorphic p forms. Thus

$$H^k(X, \mathbf{C}) = \bigoplus_{p+q=k} H^q(X, \tilde{\Omega}^p). \tag{63}$$

Also observe, that if $u \in H^{p,q}_\Delta$ then

$$du = \delta u = 0.$$

but $\delta u = (-1)^{p-q} J^{-1} \delta_a u$. So

$$\delta_a u = 0.$$

In other words, u is harmonic in the symplectic sense. Thus the Brylinski condition and hence the Strong Lefschetz property holds for Kaehler manifolds.