Solution 6

Problem 1.

- (a) For any $x_n \to x_0$ as $n \to \infty$, $|f(x_n, y)| \le g(y)$, by DCT,

  $$\lim_{n\to\infty} \int_\Omega f(x_n, y)\mu(dy) = \int_\Omega \lim_{n\to\infty} f(x_n, y)\mu(dy) = \int_\Omega \lim_{x\to x_0} f(x, y)\mu(dy)$$

  Since x_n is arbitrary, we get

  $$\lim_{x\to x_0} \int_\Omega f(x, y)\mu(dy) = \int_\Omega \lim_{x\to x_0} f(x, y)\mu(dy)$$

- (b) For any $x_n \to x_0$ and y outside a set of measure zero, by Mean Value Theorem, there exist a sequence x_n' such that $(x_n - x_n')(x_0 - x_n') \le 0$ and $\frac{f(x_n,y)-f(x_0,y)}{x_n-x_0} = f_x(x_n', y)$. Sicne $|f_x(x_n', y)| \le g(y)$. By DCT,

  $$\begin{aligned}\lim_{x_n\to x_0} \frac{\int_\Omega f(x_n, y) - \int_\Omega f(x_0, y)\mu(dy)}{x_n - x_0} &= \lim_{x_n'\to x_0} \int_\Omega f_x(x_n', y)\mu(dy) \\ &= \int_\Omega \lim_{x_n'\to x_0} f_x(x_n', y)\mu(dy) \\ &= \int_\Omega f_x(x_0, y)\mu(dy)\end{aligned}$$

  Since x_n is arbitrary, we get

  $$\frac{d}{dx} \int_\Omega f(x, y)\mu(dy) = \int_\Omega f_x(x_0, y)\mu(dy).$$

- (c) Example for (a): let $f(x, y) = \frac{1}{|x|}$ if $x \ne 0$ and $y \in (-|x|, |x|)$; 0 otherwise. Then

  $$\lim_{x\to 0} \int_\Omega f(x, y)\mu(dy) = 2 \ne 0 = \int_\Omega \lim_{x\to 0} f(x, y)\mu(dy).$$

  Example for (b): let $f(x, y) = \frac{x^3}{y^2} e^{-x^2/y}$ if $y > 0$, 0 otherwise. $\Omega = [0, 1]$. Then $f_x(x, y) = e^{-x^2/y}(\frac{3x^2}{y^2} - \frac{2x^4}{y^3})$ exists for all y.

  $$\int_\Omega f(x, y)\mu(dy) = \int_0^1 \frac{x^3}{y^2} e^{-x^2/y} dy = xe^{-x^2}$$

  $$\frac{d}{dx} \int_\Omega f(x, y)\mu(dy) = (1 - 2x^2)e^{-x^2} \ne 0 = \int_\Omega f_x(0, y)\mu(dy).$$

Problem 2.

- (a) There exists some finite constant M, such that $|f| \leq M$. Since $\lambda(I) < \infty$, $M \in L^1(I, d\lambda)$. Obviously, $|u(f, P_k)|, |l(f, P_k)| < M$, by DCT,

$$\lim_k \int_I u(f, P_k) d\lambda = \int_I u(f) d\lambda, \quad \lim_k \int_I l(f, P_k) d\lambda = \int_I l(f) d\lambda$$

Hence

$$\begin{aligned} f \text{ is Riemann integrable on } I \quad &\Longleftrightarrow \lim_k U(f, P_k) = \lim_k L(f, P_k) \\ &\Longleftrightarrow \lim_k \int_I u(f, P_k) d\lambda = \lim_k \int_I l(f, P_k) d\lambda \\ &\Longleftrightarrow \int_I u(f) d\lambda = \int_I l(f) d\lambda \end{aligned}$$

- (b) $0 \leq \lambda(\mathscr{P}) = \lambda(\bigcup \mathscr{P}_k) \leq \sum_k \lambda(\mathscr{P}_k) = 0 \Longrightarrow \lambda(\mathscr{P}) = 0$.

- (c) If f is continuous at x, $\forall \epsilon > 0$, there exist some $\delta > 0$, such that

$$\sup_{y \in [x-\delta, x+\delta]} f(y) - \inf_{y \in [x-\delta, x+\delta]} f(y) < \epsilon$$

Since P_k has limit mesh 0 and $x \notin \mathscr{P}$, there exist some N such that for all $k > N$, the subinterval created by P_k that x lies in is contained in $(x - \delta, x + \delta)$. It follows that

$$f(x) - \epsilon \leq l(f, P_k)(x) \leq f(x) \leq u(x, P_k)(x) \leq f(x) + \epsilon$$

Lettint $k \to \infty$ yields

$$f(x) - \epsilon \leq l(f)(x) \leq f(x) \leq u(x)(x) \leq f(x) + \epsilon$$

Since ϵ is arbitrary, we get $f(x) = u(f)(x) = l(f)(x)$.

Conversely, suppose $u(f)(x) = l(f)(x)$, then for every $\epsilon > 0$, we can pick some P_k such that

$$l(f)(x) - \epsilon/2 \leq l(f, P_k)(x) \leq u(f, P_k)(x) \leq u(f)(x) + \epsilon/2$$

Since $x \notin \mathscr{P}$, there exist some $\delta > 0$ such that $(x-\delta, x+\delta)$ is contained in the subinterval of P_k that x lies in. Then for $y \in (x-\delta, x+\delta)$, we have

$$l(f)(x) - \epsilon/2 \leq l(f, P_k)(t) \leq f(t) \leq u(f, P_k)(t) \leq u(f)(x) + \epsilon/2$$

Hence $|f(t) - f(x)| < \epsilon$ and f is continuous.

- (d)

$$\begin{aligned} f \text{ is Riemann integral} &\iff \int_I u(f) d\lambda = \int_I l(f) d\lambda \quad \text{by (a)} \\ &\iff \int_I (u(f) - l(f)) d\lambda = 0 \\ &\iff u(f) - l(f) = 0 \ a.s. \quad \text{since } u(f) - l(f) \geq 0 \\ &\iff f \text{ is continuous a.s.} \quad by(c) \end{aligned}$$

- (e) Since $u(f)$ is the limit of simple functions $u(f, P_k)$,$u(f)$ is measurable. f is Riemann integrable implies that $u(f) = l(f) = f$ a.s.
  From (a)
  $$\int_I f d\lambda = \int_I u(f) d\lambda = \int_I l(f) d\lambda$$
  Hence $u(f) \in L^1(I, \mathscr{B}(I), \lambda)$ and $f \in L^1(I, \overline{\mathscr{B}(I)}, \lambda)$.

Problem 3.

- (a) If f has an improper Riemann integral, then there exist $(a_n, b_n) \to (-\infty, \infty)$ as $n \to \infty$ such that f is Riemann integral on all $[a_n, b_n]$. Let $E = \{x : f \textit{ is not contiuous at } x\}$ and $E_n = E \bigcap [a_n, b_n]$. By Problem 2-(d), f is continuous a.s. on each $[-n, n]$, namely, $\lambda(E_n) = 0$.
  Hence $\lambda(E) = \lambda(\bigcup_n E_n) \leq \sum_n \lambda(E_n) = 0$.

  The converse is wrong. For example, let $f(x) = 1$. f is continuous everywhere, but f is has no improper Riemann integral.

- (b) If $f \geq 0$ has an improper Riemann integral, let $f_n = f 1_{[-a_n, b_n]} \geq 0$, where $(a_n, b_n) \to (-\infty, \infty)$ as $n \to \infty$and
  $$\int_{-\infty}^{\infty} f(x) dx = \lim_{n \to \infty} \int_{a_n}^{b_n} f(x) dx$$

Hence $f_n \in L^1(\mathbb{R}, \overline{\mathscr{B}(\mathbb{R})}, \lambda)$ by Problem2-(e).
f_n is a increasing sequence and $f = \lim_{n\to\infty} f_n$, by Monotone Convergence Theorem,

$$\int_{\mathbb{R}} f d\lambda = \int_{\mathbb{R}} \lim_{n\to\infty} f_n d\lambda = \lim_{n\to\infty} \int_{[a_n,b_n]} f d\lambda = \lim_{n\to\infty} \int_{a_n}^{b_n} f dx = \int_{-\infty}^{\infty} f(x)dx$$

and $f \in L^1(\mathbb{R}, \overline{\mathscr{B}(\mathbb{R})}, \lambda)$.

Counterexample: Letting $f = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} 1_{[n-1,n)}$, then f has an improper Riemann integral

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{n\to\infty} \int_{-n}^{n} f dx = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = ln2$$

But $f \notin L^1(\mathbb{R}, \overline{\mathscr{B}(\mathbb{R})}, \lambda)$ since $\int_{\mathbb{R}} |f| d\lambda = \infty$.