Solution 5

1.

- Prove that $\mathscr{F}$ is a σ-field:
$\emptyset$ is countable $\Longrightarrow \emptyset, \Omega \in \mathscr{F}$;
$A \in \mathscr{F} \Longrightarrow A$ or A^c is countable $\Longrightarrow A^c \in \mathscr{F}$;
If $\{A_i\}_{i=1}^{\infty} \subset \mathscr{F}$, then either each A_i is countable or there exists some A_j which is uncountable. In the first case, $\bigcup A_i$ is countable. In the second case, A_j^c is countable, so $(\bigcup A_i)^c = \bigcap A_i^c \subset A_j^c$ is countable. Hence in both case, $\bigcup A_i \in \mathscr{F}$.

- Prove that μ is a measure:
$\mu(\emptyset) = 0$ since $\emptyset$ is countable;
Suppose $\{A_i\}_{i=1}^{\infty} \subset \mathscr{F}$ are disjoint. If each A_i is countable, then $\bigcup A_i$ is countable, so $\mu(\bigcup A_i) = 0 = \sum_i \mu(A_i)$. If there exists some A_j is uncoutable, then A_j^c is countable, $\bigcup_{i \neq j} A_i \subset A_j^c$ is countable and hence for each $i \neq j$, A_i is countable. So
$$\mu(\bigcup A_i) = 1 = \mu(A_j) + \mu(\bigcup_{i \neq j} A_i) = \sum \mu(A_i)$$

- Classify the measurable functions and μ-integrable functions:
Claim: $f : \Omega \longrightarrow \mathscr{B}(\mathbb{R})$ is measuable $\Longleftrightarrow f$ is constant on some uncountable set.
Proof: "$\Leftarrow$" is obvious since if $f = a$ on uncoutable set E, then $\{f > b\}$ is countable if $b \geq a$ and $\{f > b\}^c = \{f \leq b\}$ is countable if $b < a$. In both case, $\{f > b\} \in \mathscr{F}$. Hence f is measurable.
"$\Rightarrow$": Note when f is measurable, there is atmost one point $a \in \mathbb{F}$ such that $\{f = a\}$ is uncountable, because if $a \neq b$, then $\{f = a\}$ and $\{f = b\}$ are disjoint sets in $\mathscr{F}$ and atmost one of them is uncoutable. For any n, $\{-n \leq f \leq n\} \in \mathscr{F}$ and $\bigcup_n \{-n \leq f \leq n\} = f^{-1}(\mathbb{R}) = \Omega$, so there exist some N such that $\{-N \leq f \leq N\}$ is uncoutable since Ω is uncountable. Let $[a_0, b_0] = [-N, N]$. We construct a sequence of closed intervals $[a_0, b_0] \supset [a_1, b_1] \supset [a_2, b_2] \supset ...$ by induction such that $\{a_i \leq f \leq b_i\}$ is uncountable.
$$\{a_i \leq f \leq b_i\} = \{a_i \leq f \leq (a_i + b_i)/2\} \bigcup \{(a_i + b_i) \leq f \leq b_i\}$$

is uncountable, so at least one of $\{a_i \le f \le (a_i+b_i)/2\}$ and $\{(a_i+b_i)/2 \le f \le b_i\}$ is uncoutable. If $\{a_i \le f \le (a_i+b_i)/2\}$ is uncoutable, let $[a_{i+1}, b_{i+1}] = [a_i, (a_i+b_i)/2]$; otherwise, $[a_{i+1}, b_{i+1}] = [(a_i+b_i)/2, b_i]$. Obviously, $\{a_i \le f \le b_i\} - \{a_{i+1} \le f \le b_{i+1}\}$ is countable. Then $\bigcap [a_i, b_i] = \{a\}$ and

$$\{a_0 \le f \le b_0\} - \{f = a\} = \bigcup_{i=0}^{\infty} (\{a_i \le f \le b_i\} - \{a_{i+1} \le f \le b_{i+1}\})$$

is countable, so $\{f = a\}$ is uncoutable.

- f is measurable $\Longrightarrow$ f is μ-integrable:
  If f is measurable, then there exists unique a such that $E = \{f = a\}^c$ is coutable, so $f(\Omega) = \{a\} \bigcup f(E)$ is countable. For each $x \in f(E)$, $\{f = x\} \subset E$ is countable. Hence,

  $$\int f d\mu = a\mu(E^c) + \sum_{x \in f(E)} x\mu(\{f = x\}) = a + 0 = a,$$

  which means f is μ-integrable.

2.

- By Lebesgue's Dominated Convergence Theorem, $|f_n| \le f_1$, $f_1 \in L^1(\mu)$ and $f = \lim_n f_n$ implies directly that

  $$\lim_{n\to\infty} \int_\Omega f_n(x)\mu(dx) = \int_\Omega f(x)\mu(dx)$$

- If $f_1 \in L^1(\mu)$ is omitted, take $f_n = 1_{[n,\infty)}$. Obviously, $f_n \searrow 0 = f$, but

  $$\lim_{n\to\infty} \int_\Omega f_n(x)\mu(dx) = \infty > 0 = \int_\Omega f(x)\mu(dx).$$

3.

- Let $f_n = n\log[1+(\frac{f}{n})^\alpha]$, $E_\infty = \{f=\infty\}$, $E_0 = \{f=0\}$ and $E_+ = \{0<f<\infty\}$. Then $\int_\Omega f d\mu = c > 0$ implies that $\mu(E_\infty) = 0$ and $\mu(E_+) > 0$. So $\int_\Omega f d\mu = \int_{E_+} f d\mu$.
  If $f(x) = 0$ or $f(x) = \pm\infty$, then obviously, $\lim_n f_n(x) = f(x)$. So
  $$\lim_n \int_{E_\infty} f_n d\mu + \lim_n \int_{E_0} f_n d\mu = 0, \quad \lim_n \int_\Omega f_n d\mu = \lim_n \int_{E_+} f_n d\mu.$$
  If $0 < f(x) < \infty$, then
  $$\lim_n f_n = \lim_n n^{1-\alpha} \log[1+\frac{f^\alpha(x)}{n^\alpha}]^{n^\alpha} = \log e^{f^\alpha(x)} \lim_n n^{1-\alpha} = f^\alpha(x) \lim_n n^{1-\alpha}$$
- Claim: If $\alpha \geq 1$, then $\log[1+y^\alpha] \leq \alpha y$ for all $y \geq 0$.
  Proof: Let $g(y) = \alpha y - \log[1+y^\alpha]$. Then $g(0) = 0$ and
  $$g'(y) = \alpha - \frac{\alpha y^{\alpha-1}}{1+y^\alpha} = \frac{\alpha(1+y^{\alpha-1}(y-1))}{1+y^\alpha} \geq 0,$$
  because if $y \geq 1$, then $g'(y) \geq 0$ obviously; if $0 \leq y < 1$, then $y^{\alpha-1}(1-y) < 1$, so we still have $g'(y) > 0$. Therefore, $g(y) \geq 0$ for all $y \geq 0$.
- $0 < \alpha < 1$: by Fatou Lemma,
  $$\begin{aligned} \lim_n \int_\Omega f_n d\mu &= \lim_n \int_{E_+} f_n d\mu \\ &\geq \int_{E_+} \liminf_n f_n d\mu = \int_{E_+} f^\alpha(x) \liminf_n n^{1-\alpha} \\ &= \int_{E_+} \infty = \infty \end{aligned}$$
- $\alpha = 1$: for $x \in E_+$, $\lim_n f_n(x) = f(x)$, and
  $$0 \leq f_n = n\log[1+\frac{f}{n}] \leq n\frac{f}{n} = f \in L^1(\mu).$$
  By Lebesgue's Dominated Convergence Theorem,
  $$\lim_n \int_\Omega f_n d\mu = \lim_n \int_{E_+} f_n d\mu = \int_{E_+} \lim_n f_n d\mu = \int_{E_+} f d\mu = c$$

- $\alpha > 1$: for $x \in E_+$, $\lim_n f_n(x) = f^\alpha(x) \lim_n n^{1-\alpha} = 0$, and
$$0 \leq f_n = n \log[1 + (\frac{f}{n})^\alpha] \leq n\alpha \frac{f}{n} = \alpha f \in L^1(\mu).$$
By Lebesgue's Dominated Convergence Theorem,
$$\lim_n \int_\Omega f_n d\mu = \lim_n \int_{E_+} f_n d\mu = \int_{E_+} \lim_n f_n d\mu = \int_{E_+} 0 d\mu = 0$$

4.

- (a) Let 1_{A_n}'s are measurable and non-negative. By Fatou's Lemma,
$$\begin{aligned} \mu(\liminf_n A_n) &= \int_\Omega 1_{\liminf_n A_n} d\mu = \int_\Omega \liminf_n 1_{A_n} d\mu \\ &\leq \liminf_n \int_\Omega 1_{A_n} d\mu = \liminf_n \mu(A_n) \end{aligned}$$
- (b) μ is finite implies that $\int_\Omega 1_\Omega d\mu < \infty$, namely, $1_\Omega \in L^1(\mu)$.
$$0 \leq 1_{A_n} \leq 1_\Omega \Longrightarrow 0 \leq 1_{A_n^c} = 1_\Omega - 1_{A_n} \leq 1_\Omega.$$
By Fatou's Lemma and (a)
$$\begin{aligned} \mu(\limsup_n A_n) &= \mu(\Omega - \liminf_n A_n^c) \\ &\geq \mu(\Omega) - \liminf_n \mu(A_n^c) \\ &= \limsup_n (\mu(\Omega) - \mu(A_n^c)) \\ &= \limsup_n \mu(A_n) \end{aligned}$$

5.

- (a) f is non-negative implies that μ is non-negative on $\mathscr{F}$.
Obviously, $\mu(\emptyset) = \int_\emptyset f d\nu = 0$, $\mu(\Omega) = \int_\Omega f d\nu < \infty$. If $\{A_i\} \subset \mathscr{F}$ are disjoint, then
$$\mu(\bigcup A_i) = \int_{\bigcup A_i} f d\nu = \sum \int_{A_i} f d\nu = \sum \mu(A_i).$$
Hence, μ is a finite measure.

- (b) If $\nu(B) = 0$, then for any non-negative simple funcion $h \leq f$,

  $$\int_B h d\nu \leq \| h \|_\infty \nu(B) = 0.$$

  By definition of $\int_B f d\nu$, we have $\mu(B) = \int_B f d\nu = 0$.

- (c) For each $\epsilon > 0$, there exists some non-negative simple function $h \leq f$ such that $0 \leq \int_\Omega f d\nu - \int_\Omega h d\nu \leq \epsilon/2$. Choose $0 \leq \| h \|_\infty < M < \infty$, and let $\delta = \epsilon/(2M)$. Then for any $B \in \mathscr{F}$ satisfying $\nu(B) < \delta$,

  $$\mu(B) = \int_B f d\nu \leq \int_B h d\nu + \epsilon/2 \leq M\nu(B) + \epsilon/2 < \epsilon.$$

  Therefore, μ is absolutely continuous w.r.t ν.

6. For any non-negative simple function $h(x) \leq f(x)$, $h(x-a)$ is also a non-negative simple function satisfying $h(x-a) \leq f(x-a)$. Since Borel measure is translation invariant, it is easy to show that

$$\int_{\mathbb{R}^d} h(x-a)dx = \int_{\mathbb{R}^d} h(x)dx.$$

Therefore,

$$\begin{aligned}
\int_{\mathbb{R}^d} f(x)dx &= \inf\{\int_{\mathbb{R}^d} hdx : 0 \leq h(x) \leq f(x), h \textit{ simple}\} \\
&= \inf\{\int_{\mathbb{R}^d} h(x-a)dx : 0 \leq h(x-a) \leq f(x-a), h \textit{ simple}\} \\
&\geq \inf\{\int_{\mathbb{R}^d} h'dx : 0 \leq h'(x) \leq f(x-a), h' \textit{ simple}\} \\
&= \int_{\mathbb{R}^d} f(x-a)dx
\end{aligned}$$

By symmetry, $\int_{\mathbb{R}^d} f(x-a)dx \geq \int_{\mathbb{R}^d} f(x-a-(-a))dx = \int_{\mathbb{R}^d} f(x)dx$. Hence

$$\int_{\mathbb{R}^d} f(x)dx = \int_{\mathbb{R}^d} f(x-a)dx$$