Solution 1

1. For any $\{A_n\}_{n=1}^{\infty} \subset \mathcal{F}$, let $B_n = A_n - \bigcup_{k=1}^{n-1} A_k$. $\{B_n\}_{n=1}^{\infty} \subset \mathscr{F}$ since $\mathscr{F}$ is a field and $\{B_n\}_{n=1}^{\infty}\}$ is a collection of disjoint sets. Then $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n \in \mathscr{F}$. Hence $\mathscr{F}$ is a $\sigma - filed$.

2. (a) Let $A_{n=1} = (\bigcup_{i=1}^{n} A_i)^c$. Then $\{A_1, ..., A_{n+1}\}$ gives a partition of Ω. So $\sigma\{A_1, ..., A_n\} = \{\bigcup_{i\in I} A_i : I \subset \{1, 2, ..., n+1\}\}$.

(b) Let $\mathcal{B} = \{\bigcap_{i=1}^{n} \tilde{A}_i : \tilde{A}_i \in \{Ai, A_i^c\}\} = \{B_1, ..., B_r\}$. Then $\mathcal{B}$ gives a partition of Ω. $r = \#\mathcal{B} \leq 2^n$ and 2^n can be achieved in general cases (in the sense of large probability). $\{A_1, ..., A_n\} \in \sigma(\mathcal{B})$ and $\mathcal{B} \in \sigma(\{A_1, ..., A_n\})$ give that $\sigma(\{A_1, ..., A_n\}) = \sigma(\mathcal{B})$. From (a), $\sigma\{A_1, ..., A_n\} = \{\bigcup_{i\in I} B_i : I \subset 1, 2, ..., r\}$. So $\#\mathcal{F} = 2^r \leq 2^{2^n}$ and the maximum is achieved iff $r = 2^n$.

3. Let σ_1, σ_2, σ_3, σ_4 be the $\sigma - fields$ generated by sets of (a), (b), (c), (d) respectively. Take N large enough.

(a)

$$[a, b] = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, b + \frac{1}{n}) \in \mathcal{B}(\mathbb{R}) \Longrightarrow \sigma_1 \subset \mathcal{B}(\mathbb{R})$$

$$(a, b) = \bigcup_{n=N}^{\infty} [a + \frac{1}{n}, b - \frac{1}{n}] \Longrightarrow \mathcal{B}(\mathbb{R}) \subset \sigma_1$$

(b)

$$[a, b) = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, b) \in \mathcal{B}(\mathbb{R}) \Longrightarrow \sigma_2 \subset \mathcal{B}(\mathbb{R})$$

$$(a, b) = \bigcup_{n=N}^{\infty} [a + \frac{1}{n}, b) \Longrightarrow \mathcal{B}(\mathbb{R}) \subset \sigma_2$$

(c)

$$(a, \infty) = \bigcap_{n=1}^{\infty} (a, a + n) \in \mathcal{B}(\mathbb{R}) \Longrightarrow \sigma_3 \subset \mathcal{B}(\mathbb{R})$$

$$(a, b] = (a, \infty) \bigcap (b, \infty)^c \in \sigma_3 \Longrightarrow (a, b) = \bigcup_{n=N}^{\infty} (a, b - \frac{1}{n}] \in \sigma_3 \Longrightarrow \mathcal{B}(\mathbb{R}) \subset \sigma_3$$

(d)

$$(-\infty, b) = \bigcup_{n=1}^{\infty}(b-n, b) \in \mathcal{B}(\mathbb{R}) \Longrightarrow \sigma_4 \subset \mathcal{B}(\mathbb{R})$$

$$[a, b) = (-\infty, a)^c \bigcap (-\infty, b) \in \sigma_4 \Longrightarrow \mathcal{B}(\mathbb{R}) \subset \sigma_2 \subset \sigma_4$$

4. $\mathcal{D} = \{B_r(q) : r \in \mathbb{Q}^+, q \in \mathbb{Q}^d\}$ is a countable set. For any open set $\Omega \subset \mathbb{R}^d$ and any point $x \in \Omega$, there exsit some $\epsilon > 0$ such that $B_\epsilon(x) \subset \Omega$. Taking $r \in \mathbb{Q}$, $0 < r < \frac{\epsilon}{2}$, and $q \in \mathbb{Q}^d \bigcap B_r(x)$, then $x \in B_r(q) \subset B_\epsilon(x) \subset \Omega$, $B_r(q) \in \mathcal{D}$. So, $\Omega = \bigcup_{B\in\mathcal{D}, B\subset\Omega} B$. Then union is countable since $\mathcal{D}$ is countable.

5. (i) $f^{-1}(\sigma(\mathscr{C}))$ is a $\sigma - field$: $\emptyset = f^{-1}(\emptyset) \in f^{-1}(\sigma(\mathscr{C}))$, $\Omega = f^{-1}(\Omega') \in f^{-1}(\sigma(\mathscr{C}))$, $(f^{-1}(A))^c = f^{-1}(A^c) \in f^{-1}(\sigma(\mathscr{C}))$, $\bigcup_{i=1}^{\infty} f^{-1}(A_i) = f^{-1}(\bigcup_{i=1}^{\infty} A_i) \in f^{-1}(\sigma(\mathscr{C}))$, given $A, A_i \in \sigma(\mathscr{C})$. Then $f^{-1}(\mathscr{C}) \subset f^{-1}(\sigma(\mathscr{C}))$ implies that $\sigma(f^{-1}(\mathscr{C}) \subset f^{-1}(\sigma(\mathscr{C})))$.

(ii) Let $\mathscr{G} = \{U \in \sigma(\mathscr{C}) : f^{-1}(U) \in \sigma(f^{-1}(\mathscr{C}))\}$. Then $\mathscr{G}$ is a $\sigma - field$: $f^{-1}(\emptyset) = \emptyset \in \sigma(f^{-1}(\mathscr{C})) \Rightarrow \emptyset \in \mathscr{G}$; $f^{-1}(\Omega') = \Omega \in \sigma(f^{-1}(\mathscr{C})) \Rightarrow \Omega' \in \mathscr{G}$; $B \in \mathscr{G} \Rightarrow f^{-1}(B^c) = (f^{-1}(B))^c \in \sigma(f^{-1}(\mathscr{C})) \Rightarrow B^c \in \mathscr{G}$; $\{B_n\}_{n=1}^{\infty} \subset \mathscr{G} \Rightarrow \{f^{-1}(B_n)\} \subset \sigma(f^{-1}(\mathscr{C})) \Rightarrow f^{-1}(\bigcup_{n=1}^{\infty} B_n) = \bigcup_{n=1}^{\infty} f^{-1}(B_n) \in \sigma(f^{-1}(\mathscr{C})) \Rightarrow \bigcup_{n=1}^{\infty} B_n \in \mathscr{G}$. Obviously, $\mathscr{C} \subset \mathscr{G}$. So $\sigma(\mathscr{C}) \subset \mathscr{G} \Rightarrow f^{-1}(\sigma(\mathscr{C})) \subset f^{-1}(\mathscr{G}) \subset \sigma(f^{-1}(\mathscr{C}))$.

6. (i) $\bigcap_{n=m}^{\infty} \Gamma_n$ is an increasing sequence, so

$$\mu(\liminf \Gamma_n) = \mu(\bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \Gamma_n) = \lim_{m\to\infty} \mu(\bigcap_{n=m}^{\infty} \Gamma_n) \leq \liminf_{n\to\infty} \mu(\Gamma_n)$$

, since for each m, $\mu(\bigcap_{n=m}^{\infty} \Gamma_n) \leq \liminf \mu(\Gamma_n)$.

(ii) $\bigcup_{n=m}^{\infty} \Gamma_n$ is a decreasing sequence and $\mu(\bigcup_{n=1}^{\infty}) < \infty$, so

$$\mu(\limsup \Gamma_n) = \mu(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \Gamma_n) = \lim_{m\to\infty} \mu(\bigcup_{n=m}^{\infty} \Gamma_n) \geq \limsup_{n\to\infty} \mu(\Gamma_n)$$

, since for each m, $\mu(\bigcup_{n=m}^{\infty} \Gamma_n) \geq \limsup \mu(\Gamma_n)$.

(iii) $\mu(\liminf \Gamma_n) \leq \liminf_{n\to\infty} \mu(\Gamma_n) \leq \limsup_{n\to\infty} \mu(\Gamma_n) \leq \mu(\limsup \Gamma_n)$. If $\lim_{n\to\infty} \Gamma_n$ exits, then $\mu(\liminf \Gamma_n) = \mu(\limsup \Gamma_n) = \mu(\lim \Gamma_n)$ implies that $\liminf_{n\to\infty} \mu(\Gamma_n) = \limsup_{n\to\infty} \mu(\Gamma_n)$. Hence $\lim_{n\to\infty} \mu(\Gamma_n)$ exists.

(iv) $\mu(\limsup \Gamma_n) = \lim_{m\to\infty} \mu(\bigcup_{n=m}^{\infty} \Gamma_n) \le \lim_{m\to\infty} \sum_{n=m}^{\infty} \mu(\Gamma_n) = 0$ if $\sum_{n=1}^{\infty} \mu(\Gamma_n) < \infty$.

7. (i) Let's first show that measurability of f is equivalent to: (a) $\{f \ge a\} \in \mathscr{F}$ for each $a \in \mathbb{Q}$; (b) $\{f > a\} \in \mathscr{F}$ for each $a \in \mathbb{Q}$; (c) $\{f \le a\} \in \mathscr{F}$ for each $a \in \mathbb{Q}$; (d) $\{f < a\} \in \mathscr{F}$ for each $a \in \mathbb{Q}$.

Clearly, f being measurable implies the four. Conversely, by Exericse 3, $\mathscr{B}(\mathbb{R})$ can be generated by $\{(a,\infty) : a \in \mathbb{R}\}$ or $\{(-\infty, b) : b \in \mathbb{R}\}$. Note that $\mathbb{Q}$ is dense in $\mathbb{R}$. Case (a) implies for any $a \in \mathbb{R}$, $f^{-1}(a,\infty) = f^{-1}(\bigcup_{i=1}^{\infty}(a_j,\infty)) = \bigcup_{i=1}^{\infty} f^{-1}(a_j,\infty) \in \mathscr{F}$ where $a_j \in \mathbb{Q}$ and decrese to a. By Exercise 5, $f^{-1}(\mathscr{B}((R))) \subset \mathscr{F}$. Case (b), (c), (d) are similar.

(ii)

$$\{f < g\} = \bigcup_{q\in\mathbb{Q}} (\{f < q\} \bigcap \{q < g\}) \in \mathscr{F}$$

$$\{f \le g\} = (\{g < f\})^c \in \mathscr{F}$$

$$\{f = g\} = \{f \le g\} \bigcap \{f < g\}^c \in \mathscr{F}$$

$$\{f \ne g\} = \{f = g\}^c \in \mathscr{F}$$