Solution to Midterm Exam

1.

(a) Claim: $\log(1-x) \le -x$ if $0 \le x < 1$, $\log(1+x) \le x$ if $0 \le x < \infty$.
Proof: Leg $f(x) = \log(1-x) + x$, $g(x) = \log(1+x) - x$. Then

$$f(0) = 0,\ f'(x) = \frac{-1}{1-x} + 1 = \frac{-x}{1-x} \le 0$$

for $0 \le x < 1$,

$$g(0) = 0,\ g'(x) = \frac{1}{1+x} - 1 = \frac{-x}{1+x} \le 0$$

for $0 \le x < \infty$. Hence

$$0 \le (1-\frac{x}{n})^n e^{\frac{x}{2}} 1_{(0,n)} = 1_{(0,n)} e^{n\log(1-\frac{x}{n})+\frac{x}{2}} \le e^{-\frac{x}{2}}$$

$$0 \le (1+\frac{x}{n})^n e^{-2x} 1_{(0,n)} = 1_{(0,n)} e^{n\log(1+\frac{x}{n})-2x} \le e^{-x}$$

By Dominated Convergence Theorem,

$$\begin{aligned}\lim_{n\to\infty}\int_0^n (1-\frac{x}{n})^n e^{\frac{x}{2}} dx &= \lim_{n\to\infty}\int_0^\infty (1-\frac{x}{n})^n e^{\frac{x}{2}} 1_{(0,n)} dx \\ &= \int_0^\infty \lim_{n\to\infty}(1-\frac{x}{n})^n e^{\frac{x}{2}} 1_{(0,n)} dx \\ &= \int_0^\infty e^{-x/2} dx = 2\end{aligned}$$

$$\begin{aligned}\lim_{n\to\infty}\int_0^n (1+\frac{x}{n})^n e^{-2x} dx &= \lim_{n\to\infty}\int_0^\infty (1+\frac{x}{n})^n e^{-2x} 1_{(0,n)} dx \\ &= \int_0^\infty \lim_{n\to\infty}(1+\frac{x}{n})^n e^{-2x} 1_{(0,n)} dx \\ &= \int_0^\infty e^{-x} dx = 1\end{aligned}$$

(b) For any $a > 0$,

$$\int_{(0,a)\times(0,\infty)} |\sin x e^{-xt}| dx dt = \int_{(0,a)\times(0,\infty)} |\frac{\sin x}{x} e^{-y}| dy dx < \infty$$

By Fubini's Theorem,

$$\int_0^a \frac{\sin x}{x}dx = \int_0^a \int_0^\infty \sin x e^{-xt} dt dx = \int_0^\infty \int_0^a \sin x e^{-xt} dx dt$$

$$\begin{aligned}
\int_0^a \sin x e^{-xt} dx &= \int_0^a e^{-xt} d(-\cos x) \\
&= -\cos x e^{-xt}|_0^a + \int_0^a \cos x e^{-xt}(-t) dx \\
&= 1 - \cos a e^{-at} - t \int_0^a e^{-xt} d(\sin x) \\
&= 1 - \cos a e^{-at} - t \sin x e^{-xt}|_0^a + t \int_0 \sin x e^{-xt}(-t) dx \\
&= 1 - \cos a e^{-at} - t \sin a e^{-at} - t^2 \int_0 \sin x e^{-xt} dx \\
\Longrightarrow \int_0^a \sin x e^{-xt} dx &= \frac{1}{t^2+1}(1 - \cos a e^{-at} - t \sin a e^{-at}) \\
\Longrightarrow \lim_{a\to\infty} \int_0^a \sin x e^{-xt} dx &= \frac{1}{t^2+1} \\
\textit{and } |\int_0^a \sin x e^{-xt} dx| &\le \frac{1}{t^2+1}(1 + e^{-at} + t e^{-at}) \in L^1((0,\infty))
\end{aligned}$$

By Doninated Convergence Theorem,

$$\begin{aligned}
\lim_{a\to\infty} \int_0^a \frac{\sin x}{x} dx &= \lim_{a\to\infty} \int_0^\infty \int_0^a \sin x e^{-xt} dx dt \\
&= \int_0^\infty (\lim_{a\to\infty} \int_0^a \sin x e^{-xt} dx) dt \\
&= \int_0^\infty \frac{1}{t^2+1} = \frac{\pi}{2}
\end{aligned}$$

(c) For any $t \ge 0$, $0 \le \log(1+e^t) - t = \log(\frac{1+e^t}{e^t}) \le \ln 2$. Hence for $n > 1$,

$$|\frac{1}{n}\log(1+e^{nf(x)})| \le \ln 2 + f(x) \in L^1([0,1]).$$

If $f(x) \ge 0$, $\lim_{n\to}(1+e^{nf(x)})^{1/n} = e^{f(x)}$.
If $f(x) < 0$, $\lim_{n\to}(1+e^{nf(x)})^{1/n} = 1$.

Hence, $\lim_{n\to}(1+e^{nf(x)})^{1/n} = f(x)1_{\{f>0\}}$.
By Dominated Convergence Theorem,

$$\lim_{n\to\infty}\frac{1}{n}\int \log(1+e^{nf(x)})dx = \int_0^1 \lim_{n\to\infty}\frac{1}{n}\log(1+e^{nf(x)})dx = \int f(x)1_{\{f>0\}}dx.$$

2.

(a) From the translation invariance of λ,

$$\begin{aligned}\int_{\mathbb{R}^n\times\mathbb{R}^n} |f(x-t)g(t)|\lambda(dxdt) &= \int_{\mathbb{R}^n\times\mathbb{R}^n} |f(y)g(t)|\lambda(dydt) \\ &= \parallel f \parallel_1 \parallel g \parallel_1 < \infty\end{aligned}$$

By Fubini's Theorem,

$$\int_{\mathbb{R}^n}(\int_{\mathbb{R}^n} |f(x-t)g(t)dt)|dx = \int_{\mathbb{R}^n\times\mathbb{R}^n} |f(x-t)g(t)|\lambda(dxdt) < \infty$$

Therefore, $(\int_{\mathbb{R}^n}|f(x-t)g(t|)dt) < \infty$, namely, $F_x \in L^1(\mathbb{R}^n)$, for almost every $x \in \mathbb{R}^n$.

(b) By (a),

$$\int_{\mathbb{R}^n}|\int_{\mathbb{R}^n} f(x-t)g(t)dt|dx \le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x-t)g(t)|dtdx = \parallel f \parallel_1 \parallel g \parallel_1$$

Thus $f\star g \in L^1(\mathbb{R}^n)$ and $\parallel f\star g \parallel_1 \le \parallel f \parallel_1 \parallel g \parallel_1$.

(c)

$$f\star g(x) = \int f(x-t)g(t)dt = \int f(y)g(x-y)dy = g\star f(x)$$

$$\int_{\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n} |f(x-s-t)g(t)h(s)|\lambda(dsdtdx) = \parallel f \parallel_1 \parallel g \parallel_1 \parallel h \parallel_1 < \infty$$

So for almost every $x \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} |f(x-s-t)g(t)h(s)|\lambda(dtdx) < \infty.$$

By Fubini's Theorem, for almost every $x \in \mathbb{R}^n$,

$$
\begin{aligned}
(f \star g) \star h(x) &= \int (\int f(x-s-t) g(t) dt) h(s) ds \\
&= \int (\int f(x-y) g(y-s) dy) h(s) ds \quad \textit{by letting } y = t-s \\
&= \int f(x-y) (\int g(y-s) h(s) ds) dy \\
&= f \star (g \star h)(x) \\
f \star (g+h)(x) &= \int f(x-t)(g(t)+h(t)) dt \\
&= \int f(x-t) g(t) dt + f(x-t) h(t) dt \\
&= f \star g(x) + f \star h(x)
\end{aligned}
$$

Hence $\star$ is comumutative, associate and distributive.

(d) There exist some constants $M_j > 0$ and $M > 0$ such that

$$|\partial_{x_j} f| < M_j, \quad |f(x) - f(y)| < M|x-y| \ for\ all\ x, y \in \mathbb{R}^n$$

By (a), there exist some x_0 such that $F(x_0, t) = f(x_0 - t) g(t) \in L^1(\mathbb{R}^n)$, so for each x,

$$|F(x,t) - F(x_0,t)| \leq |f(x-t) - f(x_0-t)||g(t)| < M|x-x_0||g(t)| \in L^1(\mathbb{R}^n)$$

which implies that $F(x,) \in L^1(\mathbb{R}^n)$ for each x.
Also, $g \in L^1(\mathbb{R}^n)$ implies that $g(t) < \infty$ for almost every t. Thus

$$\partial_{x_j} F(x,t) = (\partial_{x_j} f)(x-t) g(t)$$

exists for almost every t.
Furthermore,

$$|\partial_{x_j}(f(x-t)g(t))| = |(\partial_{x_j} f)(x-t) g(t)| \leq M_j |g|(t) \in L^1(\mathbb{R}^n)$$

By Problem 1(b) from Homework 6, we have

$$\partial_{x_j}(f \star g)(x) = \partial_{x_j} \int f(x-t) g(t) dt = \int (\partial_{x_j} f)(x-t) g(t) dt = (\partial_{x_j} f) \star g(x)$$

3.

(a) Define $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ by $f(x,y) = x + y$. Obviously, f is continuous, hence measurable. So $E_2 = f^{-1}(E) \in \mathscr{B}(\mathbb{R}^2)$.

(b) $\mu \star \nu(\emptyset) = \mu \times \nu(f^{-1}(\emptyset)) = \mu \times \nu(\emptyset) = 0$.
For $A_i \in \mathbb{R}$, which are disjoint, $f^{-1}(A_i) \in \mathscr{B}(\mathbb{R}^2)$ are also disjoint. Hence

$$\begin{aligned} \mu \star \nu(\bigcup A_i) &= \mu \times \nu(f^{-1}(\bigcup A_i)) = \mu \times \nu(\bigcup f^{-1}(A_i)) \\ &= \sum \mu \times \nu(f^{-1}(A_i)) = \sum \mu \star \nu(A_i) \end{aligned}$$

Therefore, $\mu \star \nu$ is a signed measure on $\mathbb{R}$.

(c) $1_{E_2}(x,t) = 1 \Leftrightarrow x + t \in E \Leftrightarrow x \in E - t \Leftrightarrow 1_{E-t}(x) = 1$.
So $1_{E_2}(x,t) = 1_{E-t}(x)$ and

$$\begin{aligned} \mu \star \nu(E) &= \mu \times \nu(E_2) = \int\int 1_{E_2}(x,t)\mu(dx)\nu(dt) \\ &= \int\int 1_{E-t}(x)\mu(dx)\nu(dt) = \int \mu(E-t)\nu(dt) \end{aligned}$$

When f is a simple function, WLOG, assuming $f = \sum_{i=1}^{N} a_i 1_{A_i}$ where A_i are disjoint, then

$$\begin{aligned} \int f d(\mu \star \nu) &= \sum_{i=1}^{N} a_i \mu \star \nu(A_i) = \sum_{i=1}^{N} \int a_i \mu(A_i - y)\nu(dy) \\ &= \sum_{i=1}^{N} \int\int f(x+y) 1_{A_i} \mu(dx)\nu(dy) \\ &= \int\int f(x+y)\mu(dx)\nu(dy) \end{aligned}$$

When f is a compactly-supported continuous positive function on $\mathbb{R}$, and μ, ν are positive measures, $\mu \star \nu$ is also positive. Choose increasing positive simple function sequence h_n such that $h_n \nearrow f$. Then

$$\begin{aligned} \int f d(\mu \star \nu) &= \lim_{n\to\infty} \int h_n d(\mu \star \nu) \\ &= \lim_{n\to\infty} \int h_n(x+y)\mu(dx)\nu(dy) \\ &= \int f(x+y) d(\mu \star \nu) \end{aligned}$$

For general compactly supported continuous function f and general signed measure μ, ν, by Hahn decomposition and decomposition for continusous functions

$$\mu = \mu^+ - \mu^-, \quad \nu = \nu^+ - \nu^-, \quad f = f_+ - f_-$$

where μ^+, ν^+ are positive measure, μ^-, ν^- are positive finite measure and f_+, f_- are compactly-supported continuous positive functions. Then we have decomposition for $\mu \star \nu$ and $\mu \times \nu$, that is

$$(\mu \star \nu) = (\mu^+ \star \nu^+ + \mu^- \star \nu^-) - (\mu^+ \star \nu^- + \mu^- \star \nu^+)$$

$$(\mu \times \nu) = (\mu^+ \times \nu^+ + \mu^- \times \nu^-) - (\mu^+ \times \nu^- + \mu^- \times \nu^+)$$

$$\begin{aligned}
\int f_+ d(\mu \star \nu) &= \int f_+ d(\mu^+ \star \nu^+) + \int f_+ d(\mu^- \star \nu^-) \\
&\quad - \int f_+ d(\mu^+ \star \nu^-) - \int f_+ d(\mu^- \star \nu^+) \\
&= \int \int f_+(x+y) \mu^+(dx) \nu^+(dy) + \int \int f_+(x+y) \mu^-(dx) \nu^-(dy) \\
&\quad - \int \int f_+(x+y) \mu^+(dx) \nu^-(dy) - \int \int f_+(x+y) \mu^-(dx) \nu^+(dy) \\
&= \int \int f_+(x+y) \mu(dx) \nu(dy) \\
\int f_- d(\mu \star \nu) &= \int \int f_-(x+y) \mu(dx) \nu(dy)
\end{aligned}$$

Hence $\int f d(\mu \star \nu) = \int \int f(x+y) \mu(dx) \nu(dy)$.

(d) For $E \in \mathbb{R}$, E_2 is symmetry, namely, $(x, y) \in E_2 \Leftrightarrow (y, x) \in E_2$. Hence,

$$\mu \star \nu(E) = \mu \times \nu(E_2) = \nu \times \mu(E_2) = \nu \star \mu(E)$$

For signed measures $\mu, \nu, \lambda : \mathbb{R} \longrightarrow (-\infty, \infty]$, let

$$E_3 = \{(x, s, t) : x + s + t \in E\}$$

$$
\begin{aligned}
(\mu \star \nu) \star \lambda(E) &= \int (\mu \star \nu)(E - t)\lambda(dt) \\
&= \int \int \mu(E - t - s)\nu(ds)\lambda(dt) \\
&= \int \int \int 1_{E_3}(x, s, t)\mu(dx)\nu(ds)\lambda(dt) \\
&= \int \int \int 1_{E_3}(x, s, t)\nu(ds)\lambda(dt)\mu(dx) \\
&= (\nu \star \lambda) \star \mu(E) \\
&= \mu \star (\nu \star \lambda)(E) \\
\mu \star (\nu + \lambda)(E) &= \mu \times (\nu + \lambda)(E_2) \\
&= \mu \times \nu(E_2) + \mu \times \lambda(E_2) \\
&= \mu \star \nu(E) + \mu \star \lambda(E)
\end{aligned}
$$

So, $\star$ is commutative, associative and distributive.

For Dirac measure δ at 0,

$$
\delta \star \mu(E) = \mu \star \delta(E) = \int \mu(E - t)\delta(dt) = \mu(E - 0) = \mu(E)
$$

(e) Lebesgue measure λ is translation invariant. $f, g \in L^1(\mathbb{R}, \lambda)$ implies that

$$
\int_{\mathbb{R}^2} |1_E(y)f(y-t)g(t)|\lambda(dydt) \leq \int_{\mathbb{R}^2} |f(x)g(t)|\lambda(dxdt) = \| f \|_1 \| g \|_1 < \infty
$$

Therefore,

$$
\begin{aligned}
\mu \star \nu(E) &= \int \mu(E - t)\nu(dt) \\
&= \int (\int 1_{E-t}(x)f(x)\lambda(dx))g(t)\lambda(dt) \\
&= \int \int 1_E(y)f(y - t)g(t)\lambda(dy)\lambda(dt) \quad \textit{by letting } x = y - t \\
&= \int 1_E(y)(\int f(y - t)g(t)\lambda(dt))\lambda(dy) \quad \textit{by Fubini's Theorem} \\
&= \int_E f \star g(y)\lambda(dy)
\end{aligned}
$$