

18.125: Spring 2008  
Take-Home Midterm

Available	Tuesday, April 1	Due	Tuesday, April 8
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1. Evaluate the following limits. Fully justify your answers!

(a)

$$\lim_{n \rightarrow \infty} \int_0^n \left(1 - \frac{x}{n}\right)^n e^{x/2} dx, \quad \text{and} \quad \lim_{n \rightarrow \infty} \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-2x} dx.$$

(b)

$$\lim_{a \rightarrow \infty} \int_0^a \frac{\sin x}{x} dx.$$

[Hint: express  $1/x$  as  $\int_0^\infty e^{-xt} dt$  and use Fubini's theorem.]

(c)

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_0^1 \log(1 + e^{nf(x)}) dx,$$

where  $f \in L^1[0, 1]$ . [Hint: what is the smallest constant  $c$  such that  $\log(1 + e^t) < c + t$  for  $t > 0$ ?]

2. Let  $f, g \in L^1(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \lambda) \equiv L^1(\mathbb{R}^n)$ , where  $\lambda$  denotes Lebesgue measure. For  $x, t \in \mathbb{R}^n$ , let  $F(x, t) = f(x - t)g(t)$ .

(a) Use Fubini's theorem to show that, for almost every  $x \in \mathbb{R}^n$ ,  $F_x \in L^1(\mathbb{R}^n)$ . [Hint: use the translation invariance of  $\lambda$ .]

(b) Define

$$f * g(x) \equiv \int F(x, t) dt = \int f(x - t)g(t) dt,$$

the **convolution** of  $f$  and  $g$ , for those  $x$  for which  $F_x \in L^1(\mathbb{R}^n)$  à la (a). Use part (a) to show that  $f * g \in L^1(\mathbb{R}^n)$ , and in fact

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1,$$

where  $\|f\|_1 = \int |f(t)| dt$ .

(c) Verify that  $*$  is commutative, associate, and distributive; that is, show that for  $f, g, h \in L^1(\mathbb{R}^n)$ ,  $f * g = g * f$ ,  $(f * g) * h = f * (g * h)$ , and  $f * (g + h) = f * g + f * h$ .

(d) Suppose that  $f$  is differentiable, and that the partial derivatives  $\partial_j f$  are all bounded functions. Show that  $f * g$  is differentiable, regardless of how rough  $g \in L^1(\mathbb{R}^n)$  is! [Hint: use 1(b) from Homework 6.]

3. (a) Let  $E$  be a Borel set in  $\mathbb{R}$ , and define  $E_2 = \{(x, y) \in \mathbb{R}^2; x + y \in E\}$ . Show that  $E_2 \in \mathcal{B}(\mathbb{R}^2)$ .
- (b) Let  $\mu, \nu: \mathcal{B}(\mathbb{R}) \rightarrow (-\infty, \infty]$  be signed measures. Define their **convolution**  $\mu * \nu$  by the formula

$$\mu * \nu(E) = \mu \times \nu(E_2).$$

Prove that  $\mu * \nu$  is a signed Borel measure on  $\mathbb{R}$ .

- (c) For  $E \in \mathcal{B}(\mathbb{R})$  and  $t \in \mathbb{R}$ , define  $E - t = \{x - t; x \in E\}$ . Show that

$$\mu * \nu(E) = \int \mu(E - t) \nu(dt).$$

Moreover, show that for compactly-supported continuous functions  $f$  on  $\mathbb{R}$ ,

$$\int f d(\mu * \nu) = \iint f(x + y) d\mu(x) d\nu(y).$$

- (d) Show that  $*$  is a commutative, associative, distributive operation on signed measures. Show also that  $\mu * \delta = \delta * \mu = \mu$  for any signed measure  $\mu$ , where  $\delta$  is the Dirac  $\delta$ -measure at 0.
- (e) Let  $f, g \in L^1(\mathbb{R}, \lambda)$ , and suppose that  $\mu(E) = \int_E f d\lambda$  and  $\nu(E) = \int_E g d\lambda$  for  $E \in \mathcal{B}(\mathbb{R})$ . Show that  $\mu * \nu(E) = \int_E f * g d\lambda$ .

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