18.125: Spring 2008
Homework 9

Available	Wednesday, May 7	Due	Friday, May 16

1. Let $\varphi \in L^\infty(\mathbb{R}^n)$, and let $p \in (0,\infty)$. Suppose there exists constants $C_p^1, \ldots, C_p^n$ such that

$$||\xi_j|^p \varphi(\xi)| \le C_j^k, \quad 1 \le k \le n, \quad \xi \in \mathbb{R}^n.$$

Show there exists a constant C_p so that

$$|\varphi(\xi)| \le \frac{C_p}{(1+|\xi|^2)^{p/2}}, \quad \xi \in \mathbb{R}^n.$$

[*Hint*: use Jensen's inequality on the finite measure space $\{1,\ldots,n\}$ with normalized counting measure.]

2. Identify the set of $p \in \mathbb{R}$ with the property that the function $\rho(\xi) = (1+|\xi|^2)^{-p/2}$ is in $L^1(\mathbb{R}^n)$.

3. Let $f(x) = e^{-|x|}$. Calculate $\hat{f}$. Use this result, together with the properties if the Fourier transform and the inverse Fourier transform, to give another proof that

$$c_t(x) = \frac{t}{\pi}\frac{1}{x^2+t^2}, \quad t > 0$$

forms a convolution semigroup.

4. Define functions g_n on $\mathbb{R}$ as follows:

$$g_n(x) = (-1)^n e^{\pi x^2} \frac{d^n}{dx^n} e^{-2\pi x^2}.$$

(a) Show that $g_n \in L^2(\mathbb{R})$, and calculate $\|g_n\|_2$.

Define $h_n = g_n/\|g_n\|_2$. The h_n are called *Hermite functions*.

(b) Show that $h_n(x) = H_n(x)\, e^{-\pi x^2}$, where H_n is a polynomial of degree n. Conclude that $h_n \in \mathcal{S}(\mathbb{R})$, the Schwartz space over $\mathbb{R}$.

The polynomials H_n are called **Hermite polynomials**.

(c) Show that h_n and h_m are orthogonal in $L^2(\mathbb{R})$ for $n \ne m$.

5. Let us formally justify the change-of-variables formula. Let $(\Omega_1, \mathscr{F}_1)$ and $(\Omega_2, \mathscr{F}_2)$ be measurable spaces, and let $F\colon \Omega_1 \to \Omega_2$ be a measurable map. Let μ be a measure on Ω_1. The **push-forward** of μ by F is the measure $\mu \circ F^{-1}$ on $\mathscr{F}_2$; that is, $(\mu \circ F^{-1})(A) = \mu(F^{-1}(A))$ for $A \in \mathscr{F}_2$ (see Homework 2, Problem **1**). If $f\colon \Omega_2 \to \mathbb{R}$ is Borel measurable, show that $F \circ f$ is in $L^1(\Omega_1, \mathscr{F}_1, \mu)$ iff $f \in L^1(\Omega_2, \mathscr{F}_2, \mu \circ F^{-1})$, in which case we have

$$\int_{\Omega_2} f\, d(\mu \circ F^{-1}) = \int_{\Omega_1} f \circ F\, d\mu.$$

6. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space. In probability theory, the sets $E \in \mathscr{F}$ are called *events*, and $\mathbb{P}(E)$ is interpreted as the "probability of occurrence" or "likelihood" of E. Two events E_1, E_2 are called **independent** if $\mathbb{P}(E_1 \cap E_2) = \mathbb{P}(E_1) \cdot \mathbb{P}(E_2)$.

(a) Suppose A and B are independent events. Show that A^c and B are independent as well.

(b) In the probability space $[0,1]$ equipped with the Borel σ-field and Lebesgue measure, find a sequence $\{E_n\}_{n=1}^{\infty}$ of events such that E_n and E_m are independent if $n \neq m$, and $\mathbb{P}(E_n) = \frac{1}{2}$. [*Hint*: Think in terms of the binary expansion of numbers in $[0,1]$.]

A measurable function $(\Omega, \mathscr{F}, \mathbb{P}) \to (\mathbb{R}, \mathscr{B}(\mathbb{R}))$ is called a **random variable**; the letters X, Y are usually used for random variables in probability theory (rather than the more customary f, g in analysis). The notion of independence is transferred to random variables through **simple random variables** (simple functions): declare that $\mathbb{1}_{E_1}$ and $\mathbb{1}_{E_2}$ are independent iff E_1 and E_2 are independent. Now extend this notion by linearity to simple random variables (i.e. if Z is independent from X and from Y, declare that Z is independent from $\alpha X + \beta Y$ for any $\alpha, \beta \in \mathbb{R}$). Finally, extend independence to all random variables by approximation: say X and Y are independent if there are sequences X_n, Y_n of simple random variables with $X_n \to X$ and $Y_n \to Y$ a.s., such that X_n and Y_m are independent for all n, m.

(c) Show that two random variables are independent in the sense described above iff for any Borel sets $B_1, B_2 \in \mathscr{B}(\mathbb{R})$, $X^{-1}(B_1)$ and $Y^{-1}(B_2)$ are independent events.

(d) Show that the strong condition in **(b)** is equivalent to the following nominally weaker one: for any $s, t \in \mathbb{R}$, $\{X \le s\}$ and $\{Y \le t\}$ are independent events.

Given a random variable X, its **distribution** μ_X is the probability measure on $\mathbb{R}$ given by pushing $\mathbb{P}$ forward by X: $\mu_X(B) = \mathbb{P}(X \in B)$ for $B \in \mathscr{B}(\mathbb{R})$. (See Homework 2, Problem **1**, and Problem **5** above.)

(e) Give an example of a sequence of non-(identically-)zero random variables $\{X_n\}_{n=1}^{\infty}$, such that X_n and X_m are independent for $n \neq m$, and such that $\mu_{X_n} = \mu_{X_1}$ for all n. (Such random variables are called i.i.d.: *independent identically distributed*. [*Hint*: if you followed the hint in part **(a)**, you can take $X_n = \mathbb{1}_{E_n}$.]

(f) Find random variables X_1, X_2, Y such that $\mu_{X_1} = \mu_{X_2}$, and yet $\mu_{X_1+Y} \neq \mu_{X_2+Y}$. This shows that the distribution of $X + Y$ depends on more than just μ_X, μ_Y; it also depends on the relationship between X and Y.

(g) If X and Y are *independent*, show that μ_{X+Y} *is* determined by μ_X and μ_Y alone; in fact, show that $\mu_{X+Y} = \mu_X * \mu_Y$, the convolution, in this case.

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