18.125: Spring 2008
Homework 8

Available	Saturday, April 26	Due	Friday, May 2

* Typo in Problem **2** corrected ($(t-z)^{-1}$ changed to $(z-t)^{-1}$) at 2pm on April 29.

1. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is measurable, and satisfies the functional equation $f(x+y) = f(x) + f(y)$ for $x, y \in \mathbb{R}$. Prove that there is a constant $a \in \mathbb{R}$ so that $f(x) = ax$ for all $x \in \mathbb{R}$. [*Hint*: first show the result holds if f is continuous. For the general case, translate the functional equation for f into a functional equation for e^{if}, and then mollify e^{if} with an approximate identity.]

2. Let $\mathbb{C}_+ = \{z \in \mathbb{C}\,;\,, \Im z > 0\}$ (the upper-half-plane). Given a probability measure μ on $\mathbb{R}$, its *Cauchy transform* (aka *Stieltjes transform*, aka *Green function*) is the function $G_\mu\colon \mathbb{C}_+ \to \mathbb{C}$ given by

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-t}\,\mu(dt).$$

(a) Show that the image of G_μ is contained in the lower-half-plane $\mathbb{C}_- = \overline{\mathbb{C}_+}$.

(b) Suppose that $\mu \ll \lambda$, given by the density $g \in L^1(\mathbb{R})$. Prove the *Stieltjes inverstion formula*:

$$g(x) = -\frac{1}{\pi}\lim_{\epsilon\downarrow 0} \Im G(x+i\epsilon), \quad \text{for almost every } x \in \mathbb{R}.$$

(c) In general, even if μ is not absolutely continuous w.r.t. Lebesgue measure, show that the Stieltjes inversion formula still holds in *weak form*. That is, show that if $f \in C_c(\mathbb{R})$ then

$$\int_{\mathbb{R}} f\,d\mu = -\frac{1}{\pi}\lim_{\epsilon\downarrow 0}\int_{\mathbb{R}} f(x)\Im G(x+i\epsilon)\,dx.$$

(d) The $\mathscr{R}$*-transform* of the probability measure μ, $\mathscr{R}_\mu$, is the function defined on the complement of some disc in $\mathbb{C}$, given by the functional equation

$$G_\mu(\mathscr{R}_\mu(z) + 1/z) = z, \quad \text{for } \Im z \text{ large}.$$

Determine the unique probability measure μ for which $\mathscr{R}_\mu(z) = z$.

3. A family $\{\mu_t\,;\, t \in [0,\infty)\}$ of probability measures on $\mathbb{R}^n$ is called a *convolution semigroup* if $\mu_t * \mu_s = \mu_{s+t}$ for $s,t \geq 0$. If $\mu_t(dx) = f_t(x)\,dx$ ($f_t \in L^1(\mathbb{R}^n)$), we also refer to $\{f_t\,;\, t \in [0,\infty)\}$ as a *convolution semigroup*. Prove the following are convolution semigroups.

(a) $\mu_t = \delta_t$, the point mass at $t \in [0,\infty)$.

(b) $g_t(x) = (2\pi t)^{-n/2} e^{-|x|^2/2t}$ on $\mathbb{R}^n$.

(c) $c_t(x) = \frac{t}{\pi}\frac{1}{x^2+t^2}$ on $\mathbb{R}$.