18.125: Spring 2008
Homework 4

Available	Saturday, March 1	Due	Friday, March 7

1. Let $B \in \mathscr{B}(\mathbb{R}^2)$. Consider the shadow (projection) of B on the x-axis:

$$P_B = \{x \in \mathbb{R}\,; \exists y \in \mathbb{R}\ (x, y) \in B\}.$$

(a) Show that P is Lebesgue measurable; that is, $P \in \overline{\mathscr{B}(\mathbb{R})}$.

(b) Give an example of a Borel set $B \in \mathscr{B}(\mathbb{R}^2)$ with the property that its shadow P_B is *not* in $\mathscr{B}(\mathbb{R})$. [*Hint*: go to the library and look at some books on"Descriptive Set Theory"; in particular, look up analytic sets.]

2. Consider the distribution function

$$\varphi(x) = \begin{cases} 0, & x < -1 \\ 1 + x, & -1 \le x < 0 \\ 2 + x^2, & 0 \le x < 2 \\ 9, & x \ge 2. \end{cases}$$

Let μ_φ be the Lebesgue–Stieltjes measure corresponding to φ. Calculate $\mu_\varphi(B_j)$ for each of the following Borel sets B_j: $B_1 = \{2\}$, $B_2 = [-\frac{1}{2}, 3)$, $B_3 = (-1, 0] \cup (1, 2)$, $B_4 = [0, \frac{1}{2}) \cup (1, 2]$, $B_5 = \{x\,;\ |x| + 2x^2 > 1\}$.

3. Let μ be a Lebesgue–Stieltjes measure corresponding to a *continuous* distribution function on $\mathbb{R}$.

(a) If A is countable, show that $\mu(A) = 0$.

(b) If B is a Borel set and $\mu(B) > 0$, must B contain and open interval?

(c) If B is a Borel set, $\mu(B) > 0$ and $\mu(\mathbb{R} - B) = 0$, must B be dense in $\mathbb{R}$?

(d) Do the answers to **(b)** and **(c)** change if μ is Lebesgue measure?

4. Let Ω be any set, and let μ be counting measure on 2^Ω ($\mu(B) = \#B$ if B is finite, and $\mu(B) = \infty$ if B is not finite). Given two functions $f, g\colon \Omega \to \mathbb{R}$, show that $f = g$ a.s.[μ] iff $f = g$. Also, show that f is μ-integrable iff $S(f) = \{x \in \Omega\,;\ f(x) \neq 0\}$ is countable and $\sum_{x \in S(f)} |f(x)| < \infty$, in which case $\int f\, d\mu = \sum_{x \in S(f)} f(x)$.

5. Exercise 3.1.14, p. 39 in Stroock.

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