18.125: Spring 2008
Homework 3

Available	Friday, February 22	Due	Friday, February 29

* Problem **3(a)** modified; updated at 9am on Thursday, February 28.

1. Does there exist an infinite σ-field with countably many members?

2. Let B be a Borel set $B \in \mathscr{B}(\mathbb{R}^n)$ and let $x \in \mathbb{R}^n$. Set $x + B = \{x + y\,;\, y \in B\}$, and $xB = \{xy\,;\, y \in B\}$.

(a) Show that $x + B$ and xB are also Borel sets. Show the same if $B \in \overline{\mathscr{B}}(\mathbb{R}^n)$ (the Lebesgue measurable sets). [*Hint*: use the "good sets" principle.]

(b) Prove that $\lambda^n(x+B) = \lambda^n(B)$; i.e. **Lebesgue measure is translation invariant**, and $\lambda^n(xB) = |x|^n\,\lambda^n(B)$. [Hint: use the "good sets" principle, and the Monotone Class Theorem.]

3. Let μ be a Borel measure on $\mathbb{R}^n$ which assigns finite measure to compact sets. In particular, μ is σ-finite.

(a) Let B be a Borel set and let $\epsilon > 0$. Show that there is an open set U and a closed set X so that $X \subseteq B \subseteq U$ and $\mu(U - X) < \epsilon$. [*Hint*: prove the statement for boxes B, then pass to all Borel sets using... you guessed it... the "good sets" principle.]

(b) Show that for any Borel set B,

$$\mu(B) = \sup\{\mu(K)\,;\, K \subseteq B,\, K \text{ compact}\} = \inf\{\mu(U)\,;\, B \subseteq U,\, U \text{ open}\}.$$

(c) Given an example of a σ-finite Borel measure ν for which the second half of **(b)** fails; that is, there is a Borel set B such that $\nu(B) < \inf\{\nu(U)\,;\, B \subseteq U,\, U \text{ open}\}$.

4. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space, and let $\mathscr{F}_\mu$ be the completion of $\mathscr{F}$ with respect to μ. For any subset $A \subseteq \Omega$, define

$$\mu^*(A) = \inf\{\mu(F)\,;\, F \in \mathscr{F},\, A \subseteq F\}, \quad \mu_*(A) = \sup\{\mu(F)\,;\, F \in \mathscr{F},\, A \supseteq F\}.$$

(a) If $A \in \mathscr{F}_\mu$, show that $\mu^*(A) = \mu_*(A) = \mu(A)$.

(b) Conversely, show that if $A \subseteq \Omega$ such that $\mu^*(A) = \mu_*(A) < \infty$, then $A \in \mathscr{F}_\mu$.

5. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space. Say that μ has an **infinite atom** A if there is a measurable set $A \in \mathscr{F}$ with $\mu(A) = \infty$ such that, for any $B \in \mathscr{F}$ with $\varnothing \neq B \subseteq A$, $\mu(B) = \infty$.

(a) If μ has an infinite atom, show that it is not σ-finite.

(b) On the other hand, suppose that μ has no infinite atoms. Furthermore, suppose $\mathscr{F}$ does not contain an uncountable collection of disjoint sets all with positive measure. Show that μ *is* σ-finite. [*Hint*: use Zorn's lemma.]