

18.125: Spring 2008
Homework 1

Available	Thursday, February 7		Due	Friday, February 15
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- * Problem 1 corrected at 9am on February 12.
- * Title corrected (18.177 \mapsto 18.125) at 1:30pm on February 14.

1. Let Ω be a set, and let \mathcal{F} be a *field* over Ω . Suppose, in addition, that if $\{A_n\}_{n=1}^\infty$ is a countable collection of *disjoint* sets in \mathcal{F} , then $\bigsqcup_{n=1}^\infty A_n$ is in \mathcal{F} . Prove that, in fact, \mathcal{F} is a σ -field.

2. Let Ω be a set, and let A_1, \dots, A_n be subsets of Ω .
 - (a) If A_1, \dots, A_n are disjoint, give a description of the minimal σ -field $\sigma\{A_1, \dots, A_n\}$ generated by the A_j .
 - (b) In general, give an upper-bound on the number of elements of $\sigma\{A_1, \dots, A_n\}$, and describe fully the conditions on the A_j under which it is achieved.

3. Show that all of the following classes of subsets of \mathbb{R} generate the Borel σ -field $\mathcal{B}(\mathbb{R})$.
 - (a) All closed intervals.
 - (b) All intervals $[a, b)$ with $-\infty < a < b < \infty$.
 - (c) All intervals (a, ∞) with $a \in \mathbb{R}$.
 - (d) All intervals $(-\infty, b)$ with $b \in \mathbb{R}$.

4. Show that every open subset of \mathbb{R}^d is a countable union of open balls. Conclude that $\mathcal{B}(\mathbb{R}^d)$ is generated by open balls. [*Hint*: consider only balls centred at points with rational coordinates.]

5. Let Ω, Ω' be two sets, and let $f: \Omega \rightarrow \Omega'$ be any function between them. Let \mathcal{C} be a class of subsets of Ω' . Prove that

$$\sigma(f^{-1}(\mathcal{C})) = f^{-1}(\sigma(\mathcal{C})).$$

[*Hint*: Use the good sets principle.]

6. Exercise 3.1.12 (p. 38) in Stroock.

7. Let (Ω, \mathcal{F}) be a measure space, and let $f, g: \Omega \rightarrow \mathbb{R}$ be measurable functions (with respect to $\mathcal{B}(\mathbb{R})$). Show that the sets $\{f = g\}$, $\{f < g\}$, $\{f \leq g\}$, $\{f \neq g\}$ are all measurable (i.e. in \mathcal{F}). Moreover, prove that measurability of f is equivalent to the statement that $\{f \geq a\} \in \mathcal{F}$ for each $a \in \mathbb{Q}$, or the analogous formulations in terms of $\{f > a\}$, or $\{f \leq a\}$, or $\{f < a\}$.

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