18.369 Problem Set 1

Due Friday, 17 February 2012.

Problem 1: Adjoints and operators

(a) We defined the adjoint $\dagger$ of operators $\hat{O}$ by:
$\langle H_1, \hat{O} H_2 \rangle = \langle \hat{O}^\dagger H_1, H_2 \rangle$ for all $H_1$ and $H_2$ in the vector space. Show that for a finite-dimensional Hilbert space, where $H$ is a column vector $h_n$ ($n = 1, \cdots, d$), $\hat{O}$ is a square $d \times d$ matrix, and $\langle H^{(1)}, H^{(2)} \rangle$ is the ordinary conjugated dot product $\sum_n h_n^{(1)} h_n^{(2)*}$, the above adjoint definition corresponds to the conjugate-transpose for matrices. (Thus, as claimed in class, “swapping rows and columns is the consequence of the “real” definition of transpose/adjoints, not the source.)

In the subsequent parts of this problem, you may not assume that $\hat{O}$ is finite-dimensional (nor may you assume any specific formula for the inner product).

(b) Show that if $\hat{O}$ is simply a number $o$, then $\hat{O}^\dagger = o^*$. (This is not the same as the previous question, since $\hat{O}$ here can act on infinite-dimensional spaces.)

(c) If a linear operator $\hat{O}$ satisfies $\hat{O}^\dagger = \hat{O}^{-1}$, then the operator is called unitary. Show that a unitary operator preserves inner products (that is, if we apply $\hat{O}$ to every element of a Hilbert space, then their inner products with one another are unchanged). Show that the eigenvalues $o$ of a unitary operator have unit magnitude ($|o| = 1$) and that its eigenvectors can be chosen to be orthogonal to one another.

(d) For a non-singular operator $\hat{O}$ (i.e. $\hat{O}^{-1}$ exists), show that $(\hat{O}^{-1})^\dagger = (\hat{O}^\dagger)^{-1}$. (Thus, if $\hat{O}$ is Hermitian then $\hat{O}^{-1}$ is also Hermitian.)

Problem 2: Completeness

(a) Prove that the eigenvectors $H_n$ of a finite-dimensional Hermitian operator $\hat{O}$ (a $d \times d$ matrix) are complete: that is, that any $d$-dimensional vector can be expanded as a sum $\sum_n c_n H_n$ in the eigenvectors $H_n$ with some coefficients $c_n$. It is sufficient to show that there are $d$ linearly independent eigenvectors $H_n$. For example, you can follow these steps:

(i) Show that every $d \times d$ Hermitian matrix $O$ has at least one nonzero eigenvector $H_1$ [... use the fundamental theorem of algebra: every polynomial with degree $>0$ has at least one (possibly complex) root].

(ii) Show that the space of $V_1 = \{ H \mid \langle H, H_1 \rangle = 0 \}$ orthogonal to $H_1$ is preserved (transformed into itself or a subset of itself) by $\hat{O}$. From this, show that we can form a $(d-1) \times (d-1)$ Hermitian matrix whose eigenvectors (if any) give (via a similarity transformation) the remaining (if any) eigenvectors of $\hat{O}$.

(iii) By induction, form an orthonormal basis of $d$ eigenvectors for the $d$-dimensional space.

(b) Completeness is not automatic for eigenvectors in general. Give an example of a non-singular non-Hermitian operator whose eigenvectors are not complete. (A $2 \times 2$ matrix is fine. This case is also called “defective.”)

(c) Completeness of the eigenfunctions is not automatic for Hermitian operators on infinite-dimensional spaces either; they need to have some additional properties (e.g. “compactness”) for this to be true. However, it is true of most operators that we encounter in physical problems. If a particular operator did not have a complete basis of eigenfunctions, what would this mean about our ability to simulate the solutions on a computer in a finite computational box (where, when you discretize the problem, it turns approximately into a finite-dimensional problem)? No rigorous arguments required here, just your thoughts.

Problem 3: Maxwell eigenproblems

(a) In class, we eliminated $E$ from Maxwell’s equations to get an eigenproblem in $H$ alone, of the form $\hat{O} H(x) = \frac{\omega^2}{\sigma} H(x)$. Show that if you instead eliminate $H$, you cannot get a Hermitian eigenproblem in $E$ except for the trivial case $\epsilon = \text{constant}$. Instead,
show that you get a generalized Hermitian eigenproblem: an equation of the form $\hat{A}E(x) = \frac{\omega^2}{c^2} \hat{B}E(x)$, where both $\hat{A}$ and $\hat{B}$ are Hermitian operators.

(b) For any generalized Hermitian eigenproblem where $\hat{B}$ is positive definite (i.e., $\langle E, \hat{B}E \rangle > 0$ for all $E(x) \neq 0$), show that the eigenvalues (i.e., the solutions of $\hat{A}E = \lambda \hat{B}E$) are real and that different eigenfunctions $E_1$ and $E_2$ satisfy a modified kind of orthogonality. Show that $\hat{B}$ for the $E$ eigenproblem above was indeed positive definite.

(c) Alternatively, show that $\hat{B}^{-1}\hat{A}$ is Hermitian under a modified inner product $\langle E, E \rangle _B = \langle E, \hat{B}E \rangle$ for Hermitian $\hat{A}$ and $\hat{B}$ and positive-definite $\hat{B}$; the results from the previous part then follow.

(d) Show that both the $E$ and $H$ formulations lead to generalized Hermitian eigenproblems with real $\omega$ if we allow magnetic materials $\mu(x) \neq 1$ (but require $\mu$ real, positive, and independent of $H$ or $\omega$).

(e) $\mu$ and $\epsilon$ are only ordinary numbers for isotropic media. More generally, they are $3 \times 3$ matrices (technically, rank 2 tensors)—thus, in an anisotropic medium, by putting an applied field in one direction, you can get dipole moment in different direction in the material. Show what conditions these matrices must satisfy for us to still obtain a generalized Hermitian eigenproblem in $E$ (or $H$) with real eigenfrequencies $\omega$.

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1Here, when we say $E(x) \neq 0$ we mean it in the sense of generalized functions; loosely, we ignore isolated points where $E$ is nonzero, as long as such points have zero integral, since such isolated values are not physically observable. See e.g. Gelfand and Shilov, Generalized Functions.