Spectral Graph Theory and its Applications September 30, 2004

Lecture 8

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8.1 Eigenvalues and Diameters of Graphs

In this lecture, I will prove two bounds relating the eigenvalues of a graph to its diameter, and another relating eigenvalues to doubling-distance. In an attempt to contradict Rafi's claim from his lecture yesterday that distance in graphs is not the right quantity to consider, I will present applications of these bounds in probability.

8.2 A first bound on diameter

Let A be the adjacency matrix of a graph. For today's lecture, we will assume that the graph is d-regular (all vertices have degree d) so the all-1's vector is an eigenvector. We could prove similar bounds without this assumption, but it will simplify our exposition. Let $M = A/2d + I/2$ be the lazy random walk matrix corresponding to A. Recall that M is positive semi-definite, and the all-1's vector is an eigenvector of eigenvalue 1.

If M^k has no zero entries, then the graph A must have diameter at most ka. Our first bound relating eigenvalues and diameter will follow from a proof that M^k has no zero entries for sufficiently large k. It is due to Chung [**?**].

Theorem 8.2.1. *Let $\mu_2 = 1 - \lambda$ be the second-largest eigenvalue of M, and δ the diameter of the corresponding graph. Then,*

$$\delta \leq \frac{\ln n}{\lambda}.$$

Before presenting the proof, I wish to recall one elementary fact from linear algebra: a symmetric matrix M can be written

$$M = V^T \Lambda V = \sum_{i=1}^{n} \mu_i v_i v_i^T,$$

where $\Lambda = \mathrm{diag}(\mu_1, \ldots, \mu_n)$ is a diagonal matrix of eigenvalues, and V is an orthonormal matrix containing the eigenvectors v_i in its columns.

Proof. Let a and b be arbitrary vertices in the graph, and let e_a and e_b be the corresponding elementary vectors. We want to show that $e_a^T M^k e_b > 0$. To this end, write

$$
\begin{aligned}
e_a^T M^k e_b &= e_a^T \left(\sum_{i=1}^n \mu_i^k v_i v_i^T \right) e_b \\
&= \sum_{i=1}^n \mu_i^k e_a^T v_i v_i^T e_b \\
&= \sum_{i=1}^n \mu_i^k v_i(a) v_i(b) \\
&= (1/n) + \sum_{i=2}^n \mu_i^k v_i(a) v_i(b),
\end{aligned}
$$

as $v_1(a) = 1/\sqrt{n}$ for all a.

We also have

$$
\begin{aligned}
\left| \sum_{i=2}^n \mu_i^k v_i(a) v_i(b) \right| &\leq \mu_2^k \sum_{i=2}^n |v_i(a)| \, |v_i(b)| \\
&\leq \mu_2^k \, \|V_{a,:}\| \, \|V_{b,:}\| \\
&\leq \mu_2^k,
\end{aligned}
$$

as each row of V has norm at most 1. □

Thus, $\mu_2^k < 1/n$ implies $e_a M^k e_b > 0$. So,

$$
\delta \leq \frac{\ln n}{\ln(1/\mu_2)} = \frac{\ln n}{\ln(1/(1-\lambda))} \leq \frac{\ln n}{\lambda}.
$$

8.3 Improving the bound on diameter

There is a natural way to improve the bound we just proved on the diameter. Let $p(X)$ be a polynomial of degree k. If we could show that $p(M)$ does not have any zero entries, then we know that the diameter of the graph is at most k. If $p(1) = 1$, then by following our previous analysis, we can show that $p(M)$ does not have any zero entries if

$$
|p(\mu_i)| < 1/n,
$$

for all $i \geq 2$. To see why this should be so, I'll go through the derivation. Let $\rho = \max_i |p(\mu_i)|$.

$$\begin{aligned} e_a^T p(M) e_b &= \sum_{i=1}^{n} p(\mu_i) v_i(a) v_i(b) \\ &= (1/n) + \sum_{i=2}^{n} p(\mu_i) v_i(a) v_i(b) \\ &\geq (1/n) - \left| \sum_{i=2}^{n} p(\mu_i) v_i(a) v_i(b) \right| \\ &\geq (1/n) - \sum_{i=2}^{n} |p(\mu_i)| \, |v_i(a)| \, |v_i(b)| \\ &\geq (1/n) - \rho. \end{aligned}$$

It is well-understood how to construct a polynomial p such that $p(1) = 1$ and $p(\mu_i)$ is small for $i \geq 2$. We begin by taking the k-th Chebyshev polynomial, T_k, which has the following properties

1. T_k has degree k.
2. $T_k(x) \in [-1, 1]$, for $x \in [-1, 1]$.
3. $T_k(x)$ is monotonically increasing for $x \geq 1$.
4. $T_k(1 + \gamma) \geq (1 + \sqrt{2\gamma})^k / 2$, for $\gamma > 0$.

We will set

$$p(x) = \frac{T_k(x/\mu_2)}{T_k(1/\mu_2)}.$$

We immediately obtain $p(1) = 1$. To bound $|T_k(x)|$ for $x \in [0, \mu_2]$, we let $\lambda = 1 - \mu_2$. So,

$$T_k(1/\mu_2) = T_k(1/(1-\lambda)) \geq T_k(1+\lambda) \geq (1 + \sqrt{2\gamma})^k / 2.$$

Thus,

$$|p(\mu_i)| \leq 2(1 + \sqrt{2\gamma})^{-k},$$

for all $i \geq 2$. It now remains to find a k such that

$$2(1 + \sqrt{2\gamma})^{-k} \leq 1/n.$$

As $(1+\epsilon)^{1+1/\epsilon} > e$ for all $\epsilon > 0$, it suffices to take

$$k = (1 + 1/\sqrt{2\lambda}) \ln(2n),$$

which implies

Theorem 8.3.1.

$$\delta \leq (1 + 1/\sqrt{2\lambda}) \ln(2n).$$

Note that this bound is much better than the previous: it depends upon $\sqrt{\lambda}$ rather than λ. Also note that we have not optimized the analysis, and the constant terms can be improved.

8.4 Doubling Distance

For a set of vertices S is a graph, we let $N(S)$ denote the union of S and all its neighbors. Similarly, we let $N^k(S) \stackrel{\text{def}}{=} N(N^{k-1}(S))$ denote all vertices at distance at most k from S. We define the doubling distance to be the least d such that for all S satisfying $|S| < n/4$,

$$\left|N^d(S)\right| \geq 2\,|S|\,.$$

We now prove:

Theorem 8.4.1. *Let G be a d-regular graph, and let λ_2 be its second Laplacian eigenvalue. Then, the doubling distance d satisfies*

$$d \leq \sqrt{\frac{8d}{\lambda_2}}.$$

In particular, this bound can be used to prove statements similar to Theorems 2. However, it has a much more natural proof.

Proof of Theorem 3. Let S have size less than $n/4$. Set $S_0 = S$, and

$$S_i = N^i(S) - N^{i-1}(S).$$

Let

$$k = \left\lceil \sqrt{\frac{8d}{\lambda_2}} \right\rceil, \tag{8.1}$$

and assume by way of contradiction that $\left|N^k(S)\right| < 2\,|S|$. We now define a test vector y by

$$y(v) = \begin{cases} k - i & \text{if } v \in S_i \text{ for } i \leq k \\ 0 & \text{otherwise.} \end{cases}$$

We clearly have

$$\sum_{(u,v)\in E} (y(u) - y(v))^2 \leq d\,|S_k|\,.$$

However, we need to normalize y to make it orthogonal to $\mathbf{1}$. So, set

$$\gamma = \sum y(u)/n \leq k\,|S_k|\,/n < k2\,|S_0|\,/n \leq k/2.$$

and

$$x(u) = y_u - \gamma.$$

The vector x is now orthogonal to $\mathbf{1}$, and

$$\sum_{(u,v)\in E} (x(u) - x(v))^2 = \sum_{(u,v)\in E} (y(u) - y(v))^2 \leq d\,|S_k| < 2d\,|S_0|\,.$$

On the other hand,

$$\sum_u x_u^2 = \sum_u (y_u - \gamma)^2 \geq \sum_{u \in S_0} (y_u - \gamma)^2 > |S_0| (k/2)^2.$$

So, we find that

$$\lambda_2 \leq \frac{x^T L x}{x^T x} < \frac{2d |S_0|}{|S_0| k^2/4} = \frac{8d}{k^2}.$$

This contradicts the value we chose for k in (8.1). □

8.5 Application: tail inequalities

The following theorem will follow quite quickly from Theorem 8.4.1:

Theorem 8.5.1. *Let $M_1, \ldots, M_m$ be matrices such that each has norm at most 1. Let X be a random matrix obtained by choosing k of the above matrices at random and adding them together. For an i, let ν_i be the median value of the i-th eigenvalue of X. Then, for all $c > 1$*

$$P\left[|\lambda_i(X) - \nu_i| > c\sqrt{32k}\right] < 2^{c-1}.$$

Ten years ago, I don't think that anyone had any idea of how to prove a theorem like this. But, it turns out to be very simple.

From Theorem 8.4.1, we first prove

Lemma 8.5.2. *Let $G = (V, E)$ be a d-regular graph with second-smallest laplacian eigenvalue λ_2. Let $f : V \to \mathbb{R}$ be a function on the vertices, and let ν denote its median value. If f satisfies $|f(u) - f(v)| < \gamma$ for all $(u, v) \in E$, then the number of vertices u for which $f(u) > \nu + c\gamma\sqrt{8d/\lambda_2}$ is at most $2^{-c}|V|$.*

The graph that we will use to prove Theorem 8.4.1 is the Johnson graph $J(m, k)$. Each vertex of this graph is associated with a subset of $\{0, 1\}^m$ of size k, and two vertices are connected by an edge if their sets differ in exactly two elements. The degree of $J(m, k)$ is $(m - k)k$, and it is not too difficult to show that $\lambda_2 = m$. For a vertex associated with set S, we set

$$f(S) = \lambda_i\left(\sum_{j \in S} M_j\right).$$

Using the Courant-Fischer theorem, we can show that if S and T differ in two elements, then

$$|f(S) - f(T)| \leq 2.$$

Theorem 8.5.1 now follows.