1. The tangent and cotangent bundle

Let $S$ be a regular surface. Then the set $TS = \{(p, w) : p \in S, v \in T_p S\}$ is called the tangent bundle of $S$. The tangent bundles comes equipped with the obvious projection map $\pi : TS \to S$ that sends $(p, v)$ to $p$.

Exercise 1:

a) Let $x : U \to V \subset S$ be a coordinate chart in $S$. Prove that there is a bijection $\phi_V : \pi^{-1}(V) \to V \times \mathbb{R}^2$. Hint, for $p \in V$ $v \in T_p S$, we can write $v = v^1 x_1 + v^2 x_2$.

b) Let $x_i : U_i \to V_i \subset S$ be two coordinate charts with $V_1 \cap V_2 \neq \emptyset$. Prove that there is an induced map $\phi_{V_1 V_2} : V_1 \cap V_2 \to GL(\mathbb{R}^2)$.

We have define the tangent bundle as a point set, and recorded some basic properties in Exercise 1. We remark further that the maps $\phi_V$ in part a) are frequently referred to as local trivializations, and the maps $\phi_{V_1 V_2}$ are transition maps.

The cotangent bundle is similarly defined as the set $T^*S = \{(p, f) : p \in S, f \in T_p^* S\}$. Here, $T_p^* S$ denotes the cotangent space at $p$, which is just the dual space to $T_p S$. It’s easily verified that the dual transition maps and local trivializations, as well as the projection map, exists.

Exercise 2:

a) For a regular surface $S \subset \mathbb{R}^3$, let $g_p$ denote the restriction of the euclidean inner product $\langle - , - \rangle$ to $T_p S$. Show that $g \in T_p^* S \otimes T_p^* S$.

b) Let $T^* S \otimes T^* S$ denote the set $\{(p, v) : p \in S, v \in T_p^* S \otimes T_p^* S\}$. Show that this set has a projection $\pi : T^* S \otimes T^* S \to S$ and local trivializations and transition maps, as in Exercise 1.

Given a local coordinate chart $x : U \to V \subset S$, it is customary to set $x_i := \frac{\partial}{\partial x_i}$, so that $\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\}$ is a basis for $T_p S$ for all $p \in V$. In many fields it is also the custom to let $\{dx^1, dx^2\}$ denote the dual basis to $\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\}$. With this notation, we note that $g_p$ can be written as $\sum_{ij} g_{ij}(p) dx^i(p) \otimes dx^j(p)$. The map $p \mapsto (p, g_p)$ is then a map of $S$ into $T^* S \otimes T^* S$ such that $\pi(g(p)) = p$. Such a map is known as a global section.

Exercise 3:

a) Define the notion of global sections of $TS$ and $T^* S$ in analogy with that of $T^* S \otimes T^* S$. 

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b) Let $\mathcal{X}(S)$ denote the space of all smooth functions on $S$. Show that for $f, g \in \mathcal{X}$ we have that $f + g, fg \in \mathcal{X}$.

c) Let $\Gamma(TS)$ denote the set of global sections of $TS$. Show that $\Gamma(TS)$ is a “vector space” over $\mathcal{X}(S)$: That is, show that for $X, Y \in \Gamma(S)$, $f \in \mathcal{X}(S)$ there are global sections $X + Y$ and $fX$ in $\Gamma(TS)$.

We remark that it is possible to redo Exercise 3 for global sections of other bundles as well. A global section $X$ of $TS$ is really just a vector field, since at each point $p \in S$, $X(p)$ is just a pair $(p, v(p))$ where $v(p)$ is an element of the tangent plane $T_p S$.

Exercise 4 (General Vector Bundles): Let $E$ be a topological space with a continuous projection map $\pi : E \to S$ (that is, $\pi^2 = \pi$, so we are identifying $S \subset E$). Assume that the set $E_p = \pi^{-1}(p)$ is a real vector space of dimension $k$ (independent of $p$). Assume also that for each $p \in S$ there is an open set $U$ containing $p$ and a bijection $\varphi_U : \pi^{-1}(U) \to U \times \mathbb{R}^k$ such that $\varphi_q := \varphi_U|_{E_q} : E_q \to \{q\} \times \mathbb{R}^k$.

The triple $(E, S, \pi)$ is known as a vector bundle, and the spaces $E$ and $S$ are called respectively the total space and base space. For $p \in S$, the set $E_p = \pi^{-1}(p)$ is the fiber over $p$.

a) Let $\varphi_U$ and $\varphi_V$ be two local trivializations at a point $p$, so that $U \cap V \neq \emptyset$, and let $\phi_{UV} : U \cap V \to GL(\mathbb{R}, k)$ be the map $\varphi_V \cdot \varphi_U^{-1}$. Each such map $\varphi_{UV}$ is called a transition map. Prove that the transition maps satisfy:

$$\varphi_{UU} = Id, \varphi_{UV} \varphi_{VV} = \varphi_{UV} \varphi_{UV} \varphi_{VV} = Id.$$

b) Show that $TS$ and $T^*S$ are vector bundles. Show that the transition maps $\phi_{UV}$ for $TS$ are the given by the Jacobian of of the change of coordinate maps, where $x(x_1, x_2) : x^{-1}(U) \to U \subset S$ and $y(y_1, y_2) : y^{-1}(V) \to V \subset S$ are local coordinate charts.

c) Given two vector bundles $E$ and $F$, show that there is a third vector bundle $E \otimes F$ with fiber $E_p \otimes F_p$. (Hint: write down the transition maps and local trivializations.)

d) Let $\Gamma(E)$ denote the set of global sections of $E$ (In this context, a global section is a smooth function $X : S \to E$ such that $\pi \circ X = Id$.) Prove that $\Gamma(E)$ is a “vector space” over $\mathcal{X}(S)$.

Exercise 5: (Inner products on Vector bundles).

a) Let $E$ be a vector bundle over $S$. Then an inner product on $E$ is a non-degenerate section $g \in \Gamma(E^* \otimes E^*)$, where $E^*$ is the vector bundle with fibers dual to the fibers of $E$. Prove that $g(p) = g_p$ is
naturally a bilinear map $E_p \times E_p \to \mathbb{R}$ for each $p \in S$

b) (This should have been done earlier actually but we’ll do it here.) Let $V$ and $W$ be vector spaces with inner products $g$ and $h$ respectively. Given bases $\{v_i\}$ and $\{w_j\}$ respectively we can write $\{g_{ij}\}$ and $\{h_{ij}\}$ the coordinates of $g$ and $h$ in the induced bases, respectively. Prove that the map $g \otimes h : V \otimes W \times V \otimes W \to \mathbb{R}$ given by

$$(A, B) \mapsto \sum_{i,j,k,l} A^{ij} B^{kl} g_{ik} h_{jl},$$

where $A = \sum_{i,j} A^{ij} v_i \otimes w_j$ and $B = \sum_{ij} B^{ij} v_i \otimes w_j$ is an inner product (Hint: Show that this map is well defined. That is, show that the defining expression for $g \otimes h$ is invariant under coordinate changes on $V$ and $W$. You may use a version of Exercise 3 in the optional exercise in problem set 6. Don’t worry about checking that this is bilinear. But do check that it is non-degenerate!)

c) Given vector bundles $E$ and $F$ over $S$ with inner products $g \in \Gamma(E^* \otimes E^*)$ and $h \in \Gamma(F^* \otimes F^*)$, prove that there is an inner product $g \otimes h \in \Gamma(E^* \otimes E^* \otimes F^* \otimes F^*)$ (Hint: Don’t do much here. I just want you to think about this as an automatic fact. I am just looking for a small rephrasing of part b))

2. The Levi-Civita connection

Given a smooth function $f \in \mathcal{X}(S)$ recall that we have already defined the differential of $f$ at $p$ to be the linear map $d_pf : T_pS \to \mathbb{R}$ given by

$$d_pf(\alpha'(0)) = \frac{d}{dt}\bigg|_{t=0} f \circ \alpha(t)$$

for a vector $\alpha'(0) \in T_pS$. In other words $d_pf \in T^*_pS$. Letting $p$ range over $s$, we then obtain a global section $df$ of $T^*S$ given by $p \mapsto (p, d_pf)$.

Exercise 6: (Gradient of a function)

a) Given a smooth function $f \in \mathcal{X}(S)$, prove that there is a vector field $\nabla f \in \Gamma(TS)$ such that, for each $p \in S$ and $v \in T_pS$, we have that

$$g_p(\nabla f, v) = d_pf(v).$$

(Hint: Recall that the inner product is an isomorphism of $T_pS$ with $T^*_pS$.)

b) Prove that the coordinates of $df$ in the basis $\{dx_1, dx_2\}$ are $\{\frac{\partial f}{\partial x_i}\}$. 

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c) Let \( g^{ij} \) denote the coordinates of the dual inner product \( g^* \) in the basis \( \{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \} \) for some choice of local coordinates (Recall that \( g^* \in \Gamma(T^*S \otimes T^*S) \)). Prove that \( \nabla f = g^{ij} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_j} \) (Here we are using Einstein notation, so that we are actually summing over all \( i \) and \( j \)).

Exercise 7 (The Levi-Civita Connection: External Characterization)

a) Let \( X, Y \in \Gamma(TS) \) be two smooth vector fields on \( S \) (Recall that \( X \) and \( Y \) are functions of the form \( p \mapsto (p, v(p)) \), where \( v(p) \in T_p(S) \)). For \( p \in S \), let \( \bar{X}_i \) and \( \bar{Y}_i \), \( i = 1, 2 \) denote extensions to of \( X \) and \( Y \) to an open set \( O \) of \( p \) in \( \mathbb{R}^3 \). That is, they are functions \( \bar{X}_i : O \rightarrow \mathbb{R}^3 \) and \( \bar{Y}_i : O \rightarrow \mathbb{R}^3 \) such that \( \bar{X}_i(q) = X(q) \) and \( \bar{Y}_i(q) = Y(q) \) for \( q \in S \cap O \). Prove that

\[
\nabla_{\bar{X}_1} \bar{Y}_1 = D_{\bar{X}_2} \bar{Y}_2
\]

b) For smooth vector fields \( X, Y \) on \( S \), define \( \nabla_X Y(p) = D_X \bar{Y}(p) - \langle D_X \bar{Y}(p), N(p) \rangle N(p) \), for smooth extensions \( \bar{X} \) and \( \bar{Y} \). Show that \( \nabla_X Y \) is well defined.

c) Show that \( \nabla : \Gamma(TS) \times \Gamma(TS) \rightarrow \Gamma(TS) \) satisfies the following relations:

i) (Leibniz Rule): \( \nabla_X (fY) = df(X)Y + f \nabla_X Y \) for \( f \in \mathcal{X}(S) \).

ii) \( \nabla_{fX} Y = f \nabla_X Y \).

iii) (Metric compatibility): \( Zg(X, Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y) \).

The object \( \nabla \) constructed above in Exercise 7 is known as the Levi-Civita connection for \( S \). It provides a way of “differentiating” vector fields defined on \( S \), in a way analogous to the way one differentiates vector fields defined on subsets of Euclidean space.

Of particular importance are the Christoffel symbols for the connection \( \nabla \). Observe that, given a choice of local coordinates, one gets, for each pair \( i \) and \( j \), an expansion

\[
\nabla \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} = \Gamma^k_{ij} \frac{\partial}{\partial x_k}.
\]

(Here again, we are using Einstein notation, so that we are summing over all values of \( k \), while \( i \) and \( j \) are fixed.) That is, the set \( \{ \Gamma^k_{ij} \} \) consists merely of the coordinates of \( \nabla \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \) in the basis \( \{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \} \). In the following exercise
shows that the Christoffel symbols are given intrinsically by the components of the metric in the local coordinates.

**Exercise 8 (Christoffel symbols in local coordinates):** Let \((g_{ij})\) denote the components of the metric \(g\) in basis \(\{dx_i\otimes dx_j\}\) induced by a local coordinate chart on \(T^*S\otimes T^*S\), and let \((g^{ij})\) denote the matrix inverse of \((g_{ij})\) (recall that \(g^{ij}\) are the components of the dual inner product \(g^*\) in the basis \(\{\frac{\partial}{\partial x_i} \otimes \frac{\partial}{\partial x_j}\}\)).

a) Prove that
\[
g_{ij,k} := \frac{\partial}{\partial x_k} g_{ij} = g(\nabla \frac{\partial}{\partial x_i}, \nabla \frac{\partial}{\partial x_j}) + g(\frac{\partial}{\partial x_i}, \nabla \frac{\partial}{\partial x_k}) + g(\frac{\partial}{\partial x_j}, \nabla \frac{\partial}{\partial x_k}).
\]

b) Prove that \(\Gamma^k_{ij} = \Gamma^k_{ji}\).

c) Prove that
\[
(2.1) \quad g_{ij,k} + g_{ki,j} - g_{jk,i} = 2\Gamma^l_{jk} g_{li}.
\]
(Hint: Use the computation from part a), and the fact that \(\nabla \frac{\partial}{\partial x_i} = \nabla \frac{\partial}{\partial x_i}\). Also, as a reminder, Einstein notation is in effect above.)

d) Use part c) to prove that
\[
\Gamma^s_{jk} = \frac{1}{2} g^{is}(g_{ij,k} + g_{ki,j} - g_{jk,i}).
\]
(Hint: Multiply each side of (2.1) by \(g^{si}\) and then sum over all \(i\).)

**Exercise 9 (The dual connection).** In this exercise, we construct a connection \(\nabla^*\) on the cotangent bundle \(T^*S\), by which we mean a map \(\nabla^* : \Gamma(TS) \times \Gamma(T^*S) \rightarrow \Gamma(T^*S)\), satisfying the analogously stated properties i) - iii) in Exercise 7.

a) For \(X,Y \in \Gamma(TS)\) and \(F \in \Gamma(T^*S)\), show that the identity
\[
d(F(Y))(X) = \nabla^*_X F(Y) + F(\nabla_X Y)
\]
defines a connection on the cotangent bundle.

b) Let \(\Gamma^s_{ij}\) be defined by \(\nabla^*_\frac{\partial}{\partial x_i} dx_j = \Gamma^s_{ij} dx_k\). Prove that
\[
\Gamma^s_{ij} = -\Gamma^s_{ik}.
\]
The Levi-Civita connections on the tangent and cotangent bundles give us a method for “differentiating ” to obtain tensors of higher rank, in a coordinate invariant manner. For simplicity, we state this exercise for tensors of type \((0, r)\). An analogous statement holds for tensors of all types however.

Exercise 10 (Tensorial derivatives): Let \(T\) be a tensor of type \((0, r)\) on a surface \(S\). Let \(X_1, \ldots, X_r, Z\) be vector fields on \(S\). Then

\[\nabla T(X_1, \ldots, X_r, Z) = ZT(X_1, \ldots, X_r) - T(\nabla Z X_1, \ldots, X_r) - \ldots - T(X_1, \ldots, \nabla Z X_r)\]

Prove that \(\nabla T\) is a tensor of type \((0, r + 1)\). (Check only that \(T\) is linear over smooth functions \(f \in \mathcal{X}(S)\). That is, check that

\[\nabla T(X_1, \ldots, X_r, fZ) = f\nabla T(X_1, \ldots, X_r, Z)\]

) Of course, this needs to be verified in all slots.

In addition to the Levi-Civita connection, there is another important notion of derivative on a surface \(S\), called Lie bracket. There is at least one important distinction between the two: The Levi-Civita connection is determined uniquely by knowledge of the metric \(g\) (Recall Exercise 7), whereas the description on of the Lie Bracket is entirely independent of the metric.

Exercise 11: Let \(X, Y \in \Gamma(TS)\) be vector fields on \(S\). Prove that there is a vector field \([X, Y]\) on \(S\) such that \([X, Y]f = YXf - XYf\).

Prove that \([-, -]\) satisfies

\[\left[X, Y\right] = -[Y, X], [[X, Y], Z] + [[Z, Y], X] + [[Y, Z], X] = 0.\]

(Hint. Choose local coordinates and write \(X = X^i \frac{\partial}{\partial x_i}, Y = Y^i \frac{\partial}{\partial x_i}\) and determine a local coordinate expression of \([X, Y]\).)

3. The Reimann tensor and Gauss’s Theorema Egregium

Before we state Gauss’s Theorema Egregium, we will take another look at the second fundamental form, this time in the context of vector bundles.

Exercise 12:(The second fundamental form): Let \(S\) be a regular surface with orientation \(N : S \to S^2\)

\[a) \text{ Let } A : \Gamma(TS) \times \Gamma(TS) \to \mathbb{R} \text{ be given by } A(X, Y) = \langle DX Y, N \rangle. \text{ Prove that } A \text{ satisfies } \]

\[fA(X, Y) = A(fX, Y) = A(X, fY)\]

\[c) \text{ Prove that } A(X, Y) = -g(dN(X), Y). \text{ (In particular, the value of } A(X, Y) \text{ is determined by the pointwise value of } X \text{ and } Y)\].
As a consequence of Exercise 11, we have that the second fundamental form $A$ is a global section of $T^*S \otimes T^*S$.

The mean and gauss curvatures of a regular surfaces $S$ are respectively the negative trace and determinant of $d_pN : T_pS \to d_pS$. Gauss’s Theorema Egregium states that the gauss curvature of a surface $S$ is invariant under isometries. An isometry is a map $\varphi : S_1 \to S_2$ that is “metric preserving” in the following sense: $g_1_p(v,w) = g_2_{\varphi(p)}(d\varphi_p(v),d\varphi_p(w))$ for all $p \in S$. It should be noted that another way of writing this is that $g_1 = \varphi^*g_2$, where $\varphi^*g_2$ is the metric on $S_1$ defined by the previous equation. It is called the pull-back metric.

Part of the strength of Gauss’ Theorema Egregium lies in fact that it unites an “intrinsic” and “extrinsic” perspective. The notion of isometry can be thought of as an intrinsic one int he following sense: Suppose you are moving along a path $\alpha_1$ in the surface $S_1$ and let $\alpha_2$ be the curve $\varphi \cdot \alpha_1$. Then from your perspective, it should feel exactly the same moving along $\alpha_2$ as it does moving along $\alpha_1$ since there is no “stretching” felt under along the curve. Thus, in an intrinsic sense, isometric surfaces are equivalent.

Extrinsically, for example viewed from a distant point in $\mathbb{R}^3$, isometric surface can be quite different. In particular, their second fundamental forms need not be closely related at all. For example, it is easy to see that the flat plane and the cylinder are isometric. Clearly, however, the second fundamental forms are very distinct.

What the Theorema Egregium says is that, although the full second fundamental forms of isometric surfaces can behave very differently, their determinants are identical. The Theorem will be an easy consequence of some properties of the Riemann curvature tensor.

**Exercise 13:** Let $\mathcal{R} : \Gamma(TS) \times \Gamma(TS) \times \Gamma(TS) \times \Gamma(TS) \to \mathbb{R}$ be given by

$$\mathcal{R}(X,Y,Z,W) = g(\nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z - \nabla_{[X,Y]} Z, W).$$

Prove that $\mathcal{R}$ is a $(0,4)$ tensor on $S$.

**Exercise 14:** Prove that tensors depend only on the pointwise value of their inputs

One can consider the Riemann curvature tensor to be a type of measure of the failure of the second derivative of vector fields to commute. In the following, we derive a relation between $\mathcal{R}$ and the second fundamental form. Note that we can write

$$D_X Z = \nabla_X Z + A(X,Z)N$$

$$D_Y D_X Z = D_Y(\nabla_X Z) + (D_Y A(X,Z))N + A(X,Z) dN(Y).$$
For any smooth vector fields on Euclidean space we have that \( D_Y D_X Z - D_X D_Y Z - D_{[X,Y]} Z = 0 \). Thus, projecting onto the tangent plane of \( S \) and taking orthonormal coordinates at a point \( p \) we get that

\[
0 = g(\nabla_{\frac{\partial}{\partial x_1}} \nabla_{\frac{\partial}{\partial x_2}} \frac{\partial}{\partial x_1} - \nabla_{\frac{\partial}{\partial x_2}} \nabla_{\frac{\partial}{\partial x_1}} \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})
+ A(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1}) A(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_2}) + A^2(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})
= \mathcal{R}(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}) - \det A
\]

Gauss’s Theorema Egregium then follows by noting that \( \mathcal{R}(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}) \) can be written in terms of Christoffel symbols and their derivatives.