# Minkowski's and Keller's Cube-Tiling Conjectures

## 1 Introduction

This lecture explores Minkowski's and Keller's cube-tiling conjectures. One can view it as a case study of one interesting thread of mathematical research, winding through three centuries and several diverse areas of mathematics. Minkowski's conjecture arose originally in 1896 while Minkowski was considering a problem in Diophantine approximation. Investigating when an inequality concerning Diophantine approximations was tight led Minkowski to formulate his conjecture, a problem in high-dimensional geometry. He originally stated it as a theorem, the proof of which he promised to publish at a later date. Eleven years later, he stated it as a open problem, leaving one to suspect that he discovered a hole in his original proof. In 1938, in his PhD thesis, Hajós turned Minkowski's conjecture into a problem in the theory of abelian groups. Three years later, he solved it using algebra.

Meanwhile, in 1930, Keller generalized Minkowski's cube-tiling conjecture to what has come to be known as Keller's cube-tiling conjecture. Oskar Perron published a paper proving Keller's conjecture in six and fewer dimensions in 1940. (He simultaneously published a paper proving Minkowski's paper in eight and fewer dimensions.) Even though the original motivation for the problem had disappeared, it appealed to some mathematicians as an intrinsically interesting question. Corrádi and Szabó, in two papers in the late 1980's, turned Keller's conjecture into a combinatorial problem, and attempted to find a counterexample via computer search. Jeff Lagarias heard about it from Victor Klee, who gave a lecture on interesting geometry problems at a meeting in Oberwolfach, Germany. Jeff told me about the problem and we proceeded to work on it together. We found a counterexample in twelve dimensions in 1992, which we were promptly able to reduce to ten dimensions, leaving only three dimensions (seven, eight, and nine) for which the conjecture was open. In 2000, John Mackey found an eight-dimensional counterexample.

What are these conjectures? Minkowski's conjecture is

**Conjecture 1 (Minkowski)** *Every lattice tiling of $\mathbb{R}^n$ by unit cubes contains two cubes that meet in an $n-1$ dimensional face.*

Keller's conjecture is nearly the same; we merely drop the word *lattice*:

**Conjecture 2 (Keller)** *Every tiling of $\mathbb{R}^n$ by unit cubes contains two cubes that meet in an $n-1$ dimensional face.*

Except for the word "lattice," most of this terminology is probably self-evident, but let me proceed to explain it anyway. A tiling of $\mathbb{R}^n$ by unit cubes is a set of cubes such that every point in $\mathbb{R}^n$ is covered by one of the cubes, and such that the interior of no two cubes overlap. It turns out (we will not prove this in these notes) that we can assume without loss of generality that all the cubes are aligned parallel to the coordinate axes. Two cubes thus meet in an $n-1$ dimensional face if their centers differ by exactly 1 in one coordinate, and are equal in the other $n-1$ coordinates. This leaves only the definition of *lattice tiling*. For this, we will make a digression into the theory of lattices. This was one of the areas of Minkowski's research, and we will need to know a number of facts about lattices before we can proceed with the motivation for Minkowski's conjecture.

# 2 Lattices

When Minkowski posed his question about cube tilings, he was investigating both lattices and Diophantine approximation. It is thus not surprising that he used lattices in the proofs of his theorems about Diophantine approximations. Indeed, his conjecture arose from exactly this interaction. Informally, a lattice in $\mathbb{R}^n$ is a "homogeneous" set of isolated points that includes the origin.[1] That is, a lattice looks the same from every point of the lattice. The formal definition is:

**Definition 1** *A lattice in $\mathbb{R}^n$ is a set of points that is a group under vector addition, and such that there is some minimum distance $d_{\min}$ between any two points of the lattice.*

The condition that the lattice points form a group is exactly the condition that the lattice looks the same from an arbitrary point as it does from the origin. To translate a lattice point to the origin, we subtract the coordinates of this point from every point in the lattice. For this process to yield the same lattice, the set of lattice points must be closed under coordinate-wise addition and subtraction, and therefore must form a group. The condition that there is a minimum distance between every pair of points is equivalent to the condition that each lattice point is isolated. Because the lattice looks the same from every point, the minimum distance between any pair of points is the same as the minimum distance from the origin to another lattice point.

We now prove a theorem about lattices

**Theorem 1** *If $L$ is a lattice in $\mathbb{R}^n$, then there is a set of linearly independent vectors $v_1$, $v_2$, ... $v_k$ with $k \leq n$ such that $L$ consists of all points of the form*

$$a_1 v_1 + a_2 v_2 + \ldots + a_k v_k$$

*with $a_1$, $a_2$, ... $a_k$ integers.*

[1] This is a geometric lattice — there is a completely different mathematical object called a lattice that arises in logic and combinatorics.

Such a set is called a *basis* for the lattice $L$. The number of linearly independent basis vectors $k$ does not depend on the choice of this basis; we will call this the *intrinsic dimension* of the lattice.
**Proof:** We will prove this by induction. Suppose that it is true for lattices in $\mathbb{R}^{n-1}$. Consider a lattice $L \subset \mathbb{R}^n$. Let the shortest vector in this lattice be $v_s$. Now, project all the vectors onto the $n-1$ dimensional subspace perpendicular to $v_s$. If we choose the coordinates of $\mathbb{R}^n$ so that $v_s = (0, 0, 0, \ldots, 0, |v_s|)$, then this projection operation is performed by throwing away the last coordinate of every vector. Let us call this projected set of points (in $n-1$ dimensions) $L'$. We claim that $L'$ forms an $n-1$ dimensional lattice.

First, it is an easy exercise in geometry to show that the distance between every pair of points in $L'$ is at least $|v_s| \cdot \sqrt{3}/2$, as otherwise there would be a pair of points in $L$ with distance between them less than $v_s$. Second, the points of $L'$ form a group under vector addition, since first projecting onto the subspace and then adding two vectors gives the same result as first adding the two vectors and then projecting onto the subspace (this is clear if we consider the lattices in the coordinate system described above). Thus, $L'$ is an $n-1$ dimensional lattice. By the induction hypothesis, we can thus find a basis $v_1$, $v_2$, ..., $v_{k-1}$, of $L'$, with $k-1 \leq n-1$. If we add $v_k = v_s$ to this basis, we obtain a basis for $L$. $\square$

For a lattice in $\mathbb{R}^n$ with basis $v_1$, $v_2$, ..., $v_n$ as a basis, a *unit cell* is the parallelepiped formed by vectors $v_1$, $v_2$, ..., $v_n$. The volume of this unit cell is the absolute value of the determinant of the matrix with $v_k$ as its $k$'th row. While for any given lattice, there are many different choices of basis, and thus many different unit cells, the volume of a unit cell is determined solely by the lattice, independent of the choice of basis. For a sufficiently large ball, the number of lattice points in the ball is to first order the ratio of the volume of the ball to the volume of a unit cell.

We now have enough facts about lattices that we can prove a theorem of Minkowski. We will need this theorem to describe the motivation for Minkowski's conjecture.

**Theorem 2 (Minkowski)** *Let $C$ be a bounded convex body in $n$-dimensional space. Assume that its volume is greater than $2^n d$ and that it is centrally symmetric about the origin. Let $L$ be an $n$-dimensional lattice with determinant $d$. Then $C$ contains a point of $L$ other than the origin.*

Here, *convex* means that if two points are in $C$, every point between them is in $C$; *bounded* means that no point of $C$ is more than distance $D$ from the origin for some large $D$; and *centrally symmetric* about the origin means that if $x \in C$, then $-x \in C$.
**Proof:** Let us shrink $C$ by a factor of 2 in every dimension to get a smaller convex body $K$. Since $\mathrm{vol}(K) = \mathrm{vol}(C)/2^n$, $K$ has volume greater than $d$. Now, consider placing a copy of $K$ centered at every lattice point of $L$. For any $\epsilon$, we can find a sufficiently large ball $B$, such that the sum of the volumes of the copies of $K$ contained in $B$ is at least $(1-\epsilon)\mathrm{vol}(B)\mathrm{vol}(K)/\det(L)$, which for small $\epsilon$ is greater than $\mathrm{vol}(B)$. Thus, there must be two copies of $K$ which intersect, as there is not enough room in $B$ for them all to be disjoint. This gives us two points $k_1, k_2 \in K$ and two

points $v_1, v_2 \in L$ such that $v_1 + k_1 = v_2 + k_2$. Now, $k_2 - k_1 = v_1 - v_2 \in L$. But since $K$ is centrally symmetric, $-k_1$ is a point in $K$, and since $K$ is convex, the point halfway between $k_2$ and $-k_1$, namely $\frac{1}{2}k_2 - \frac{1}{2}k_1$, is in $K$. Because $C$ is $K$ expanded by a factor of 2, $k_2 - k_1 \in C$. But we showed earlier that $k_2 - k_1$ is a non-zero lattice point of $L$, proving the theorem. $\square$

Now, we can let the volume of $C$ be equal to $2^n d$, instead of greater than it, and add the condition that $C$ is closed, to obtain a corollary:

**Corollary 2.1 (Minkowski)** *Let $C$ be a centrally symmetric, bounded, closed convex body in $n$-dimensional space with volume $2^n d$. Let $L$ be an $n$-dimensional lattice with determinant $d$. Then $C$ contains a point of $L$ other than the origin.*

The proof of this comes from considering the sequence of convex bodies $C_{1+\epsilon}$, for $\epsilon > 0$, which we define to be $C$ expanded by a factor of $1 + \epsilon$. Theorem 2 applies to all of these $C_{1+\epsilon}$, so every $C_{1+\epsilon}$ contains non-zero points of the lattice $L$. Now, because $C_{1+\epsilon'} \subset C_{1+\epsilon}$ whenever $\epsilon' < \epsilon$, and there are only a finite number of lattice points in any of them, there is some lattice point contained in all of them. And now, because $C$ is closed, this lattice point must also be contained in $C$. $\square$

# 3 The motivation for Minkowski's conjecture

We now have enough background to give the motivation for Minkowski's conjecture. Minkowski was investigating the approximation of real numbers by fractions, a topic which was touched upon in the previous lecture in this series on continued fractions.

**Theorem 3 (Minkowski)** *Let $r$ be a real number and $t$ a positive integer. There are integers $x_1$ and $x_2$ such that*

$$\left| r - \frac{x_1}{x_2} \right| \leq \frac{1}{tx_2} \quad \text{and} \quad x_2 \leq t.$$

**Proof:** Consider a lattice in $\mathbb{R}^2$ with basis $(1, 0)$ and $(-r, 1)$ This lattice has a unit cell with area

$$\left| \det \begin{pmatrix} -r & 1 \\ 1 & 0 \end{pmatrix} \right| = 1.$$

Let $C$ be a rectangle of width $2/t$ and height $2t$ centered on the origin. This body has area 4, which is $2^2 \cdot 1$, so we can apply Corollary 2 to show that there is a lattice point within $C$. Let this lattice point be

$$x_2(-r, 1) + x_1(1, 0) = (-x_2 r + x_1, x_2)$$

Then, since this lattice point is in $C$, we have

$$|x_2| \leq t$$

and

$$|-x_2r + x_1| \le \frac{1}{t}.$$

Dividing the second equation by $x_2$, we obtain

$$\left|r - \frac{x_1}{x_2}\right| \le \frac{1}{t|x_2|},$$

which is what we wanted. □

Minkowski then extended this proof to show that you can simultaneously approximate two real numbers $r$ and $s$.

**Theorem 4 (Minkowski)** *Let $r, s$ be two real numbers and $t$ some real number greater than 1. There are integers $x_1$, $x_2$ and $x_3$, with $x_3 \le t^2$, such that*

$$\left|r - \frac{x_1}{x_3}\right| \le \frac{1}{tx_3} \quad \text{and} \quad \left|s - \frac{x_1}{x_3}\right| \le \frac{1}{tx_3}.$$

The proof is almost the same as for the case of approximating one real number.
**Proof:** Consider the lattice $L$ in $\mathbb{R}^3$ with basis $(t, 0, 0)$, $(0, t, 0)$, $(-rt, -st, 1/t^2)$. This has determinant

$$\det \begin{pmatrix} t & 0 & 0 \\ 0 & t & 0 \\ -rt & -st & 1/t^2 \end{pmatrix} = 1.$$

Now, consider the cube $C$ with vertices $(\pm 1, \pm 1, \pm 1)$. This has volume $8 = 2^3$. Thus, by Corollary 3, there is a non-zero point of $L$ in the cube. We let this point of the lattice be

$$x_1(t, 0, 0) + x_2(0, t, 0) + x_3(-rt, -st, 1/t^2) = (x_1t - x_3rt, x_2t - x_3rt, x_3/t^2).$$

We thus have

$$|x_1 - x_3r|t \le 1 \quad |x_2 - x_3s|t \le 1 \quad |x_3/t^2| \le 1.$$

This gives (since the first two inequalities show that $x_3 \neq 0$, and if $x_3$ is negative we can change the signs of all the $x_i$ to make it positive)

$$\left|r - \frac{x_1}{x_3}\right| \le \frac{1}{tx_3} \le \frac{1}{x_3^{3/2}}$$

and similarly

$$\left|s - \frac{x_2}{x_3}\right| \le \frac{1}{tx_3} \le \frac{1}{x_3^{3/2}}.$$

The same proof technique works in higher dimensions to give a bound on how well one can simultaneously approximate $k$ real numbers with fractions.

Now, Minkowski was interested in when Theorem 4 held with equality, i.e., when it ceased to be true if all the $\leq$ signs were replaced with $<$ signs. Theorem 4 can only hold with equality if the non-zero lattice points given by Corollary 2.1 all lie on the boundary of the cube $C$. Thus, he needed to know when a lattice $L$ with determinant 1 has no non-zero lattice points in the interior of the cube $C$ with vertices $(\pm 1, \pm 1, \pm 1, \ldots \pm 1)$. Now, let us assume that we have a lattice $L$ with determinant 1 has no points in the interior of the cube $C$. Then, for every point in the interior of the cube $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \ldots \pm \frac{1}{2})$, the closest lattice point of $L$ is the origin. Consider putting a unit cube around every lattice point of $L$. Since $L$ is a lattice, it looks the same from every point, so the closest lattice point to any point inside one of the cubes must be the center of the cube. The interiors of any two of these cubes thus cannot intersect, as otherwise the center of the cubes would not be the closest lattice points to every point in the interior of these cubes. Further, the unit cell of the lattice has volume 1, which is the same as the volume of a unit cube. Thus, the unit cubes must exactly tile $\mathbb{R}^n$. Minkowski identified one class of lattices where the points are only on the exterior of the unit cube. In three dimensions, this class is those lattices generated by basis vectors

$$(1, \alpha, \beta), \quad (0, 1, \gamma), \quad (0, 0, 1)$$

when $\alpha, \beta, \gamma \in \mathbb{R}$ as well as lattices obtained by permuting the coordinates of this lattice. Minkowski proved that in three dimensions, these were the only such lattices. The generalization to $n$ dimensions of this class of lattices is straightforward; up to permutation of the coordinates, the lattice is generated by $n$ basis vectors of the form

$$(0, 0, \ldots, 0, 1, \alpha_{k,k+1}, \alpha_{k,k+2}, \ldots, \alpha_{k,n}),$$

where the $k$'th basis vector starts with $k-1$ 0's, and where the $\alpha_{k,l}$ are arbitrary real numbers Because a vector with $n-1$ 0's and one 1 is contained in this basis, two of the 3-dimensional cubes in the tiling constructed above share a face. Minkowski realized that if he could prove that every $n$-dimensional lattice tiling by unit cubes contained two vectors that shared a face, then by induction he could prove that all lattice tilings of $\mathbb{R}^n$ by cubes fell into the class of tilings he had found. He was able to prove this theorem for three dimensions, but not for higher dimensions.

# 4 Hajós's group theoretic version of Minkowski's conjecture

First, we show that we can assume w.l.o.g. that all the vectors of the lattice in a counterexample to Minkowski's conjecture have rational coordinates. If we can ensure that all the vectors in a basis have rational coordinates, then we will be done. Let us consider a counterexample with some vectors having irrational coordinates. We must show that we can perturb all the coordinates in a basis for the lattice to give them rational coordinates, and still ensure that it is a counterexample to Minkowski's conjecture. To ensure it is a counterexample, all we need is that no cube meets the cube centered at the origin in an $n-1$ dimensional face. Since only a finite number of cubes touch the cube at the origin, a small enough perturbation will preserve the property of being a counterexample to Minkowski's conjecture, and make all the coordinates rational.

We now need to introduce some group theory. Let us assume we have a lattice $L$, all of whose points have rational coordinates. By choosing the least common denominator of all the coordinates of the basis vectors, we can assume that all the cubes are centered at points with coordinates which are integer multiples of some integer $r$. Now, consider the lattice $L'$ generated by $(1/r, 0, 0, \ldots, 0)$, $(0, 1/r, 0, 0, \ldots, 0)$, $(0, 0, 1/r, 0, \ldots, 0)$, $\cdots$, $(0, 0, 0, 0, \ldots, 1/r)$, The lattice $L$ is a sub-lattice of this lattice. Consider the set of translations by elements in $L'$. These form an abelian group. We will consider two of these translations to be equivalent if there is a translation by an element of $L$ that takes one to the other; this is equivalent to saying that the coordinate-wise difference between them is in the lattice $L$. The equivalence classes formed in this way make an abelian group. This group is called the factor group $L'/L$. The number of elements in this group is the ratio of the volume of the unit cells of $L$ and of $L'$. We let $\bar{0}$ be the identity element of the group, and we will represent group operations by addition rather than multiplication (the usual convention).

For the next part of the argument, instead of forming a cube tiling by placing the centers of the unit cubes on lattice points of $L$, we will need to put a corner of each of these unit cubes on lattice points of $L$ (in the plane, the lower left corner; in higher dimensions, the corner with smallest entry in each coordinate dimension). Let $a_i$ be the point of $L'$ with $i$'th coordinate $1/r$ and the other coordinates $0$. Because translating a unit cube by all the vectors of $L$ tiles space, the elements of $L'/L$ correspond to the lattice points in one of these unit cubes. We can thus express every element of the factor group $L'/L$ as a product sum $x_1 a_1 + x_2^{a_2} + \ldots + x_n a_n$ where the $x_i$ are integers with $0 \leq x_n < r$. There will be two cubes that meet in an $n-1$ dimensional face if and only if $ra_i = \bar{0}$ for one of the $a_i$. This gives us Hajós' reformulation of Minkowski's conjecture.

**Theorem 5 (Hajós)** *If $a_1$, $a_2$, …, $a_n$ are elements of an abelian group $G$ such that every element of $G$ is uniquely expressible as $x_1 a_1 + x_2 a_2 + \ldots + x_n a_n$ where the $x_i$ are integers with $0 \leq x_1, x_2, \ldots, x_n < r$, then one of the elements $a_i$ must satisfy $ra_i = \bar{0}$.*

In fact, Hajós formulated the slightly more general conjecture where we may have a different order $r_i$ for each $a_i$; we require $0 \leq x_i < r_i$, and the conclusion is that for some $i$, $r_i a_i = \bar{0}$. Whereas Hajós had originally assumed that this group-theoretic formulation would be fairly easy to prove, this was not so. It took him three years: he proved his theorem in 1941, and his proof was so algebraic that it obscured any connection with the original geometric formulation. In Stein and Szabó's book, he is quoted as saying "Yes, I had a proof, but I couldn't see what was really happening."

# 5 Keller's conjecture

We now turn our attention to Keller's conjecture. Keller suggested that the condition in Minkowski's conjecture that the cube tiling form a lattice was actually unnecessary. He thus conjectured that any tiling of $\mathbb{R}^n$ by unit cubes contains two cubes that meet in an $n-1$ dimensional face.

Keller's conjecture was proved in dimensions 6 and lower by Perron. His paper performs an extensive case analysis which he carries out in great detail. Before he begins the proof, he remarks (in German; I translate it here)
*Since our geometrical intuition of three-dimensional space entices us all too easily to over-hurried analogistic reasoning in n-dimensional space, we wish to do without difficult geometrical terms and make everything arithmetic; even seemingly completely trivial theorems which we will need are to be provided real proofs for the sake of caution.*
This cautionary note was probably inspired by his opinion of previous papers on Keller's conjecture. In his review of the literature, his comments on three of these papers are "not always comprehensible;" "for the most part incomprehensible to me;" and "very sketchy and seemingly incomprehensible to me." This last was a short paper of Keller's which claimed to prove Keller's conjecture in dimensions up to 6, and Minkowski's conjecture in dimensions up to 8, the very cases which Perron proves in his two papers. In the homework, you will see how Perron's proof techniques work in dimension 3.

We now describe how Szabó changed Keller's conjecture into a combinatorial problem, and how they tried to attack the problem. We first need a lemma.

**Lemma 1** *If there is an $n$-dimensional cube tiling such that no two cubes meet in a $k$ dimensional face, then there is such a cube tiling with period 2 in every coordinate direction.*

This lemma shows that to look for $n$-dimensional counterexamples to Keller's conjecture, we need only consider cube tilings with period 2, i.e., if a point $x$ is in the cube, then every point

$$x + (2a_1, 2a_2, \dots, 2a_n)$$

is in the cube, where $a_i$ are integers.
**Proof:** Consider a cube tiling in $\mathbb{R}^n$. Let $K$ be the set of cubes with centers $(x_1, x_2, \dots, x_n)$, with $0 \le x_i < 2$. We will show that repeating the cluster of cubes $K$ with period 2 generates a tiling of $\mathbb{R}^n$, and that there are two cubes that meet in a $k$-dimensional face in the new tiling if and only if there are two cubes that meet in a $k$-dimensional face in the original tiling.

We first need to show that the new set of cubes generates a tiling of $\mathbb{R}^n$. To do this, we will make the coordinates periodic one at a time. For the first stage, we take the cubes which have coordinates $x_1$ with $0 \le x_1 < 2$, and consider this set of cubes. Consider any line parallel to the $x_1$ axis that does not run along the boundary of any cubes in the original cube tiling. I claim that this line must pass through exactly two cubes, and that these cubes intersect the line in an interval of length 2. Why is this? The intersection of the line with the original cube tiling is a one-dimensional cube tiling, and there is only essentially one of these: the division of the line into unit intervals. If we choose the cubes in this one-dimensional cube tiling with centers $x_1$ satisfying $0 \le x_1 < 2$, it is not hard to see that we must obtain exactly two unit intervals. Thus, if we take the set of cubes with first coordinate $0 \le x_1 < 2$, we can use this set to make a cube tiling of $\mathbb{R}^n$ that is periodic with period 2 in the first coordinate. If we perform this procedure one coordinate at a time, the only cubes that are never replaced are those having $0 \le x_i < 2$ for all coordinates $x_i$, so this set of cubes must tile $\mathbb{R}$ with period 2 in all coordinate dimensions. For the set of cubes to have the right volume to tile $\mathbb{R}^n$, there must be $2^n$ of them.

We now must show that if two cubes in the new tiling meet in a full $k$ dimensional face, then two cubes in the original tiling did. Two cubes meet in a $k$ dimensional face if their coordinates are the same in $n - k$ dimensions, and differ by exactly one in in the remaining $k$ dimensions. Suppose we have two such cubes. For each of these two cubes, let us consider the cube in the original tiling that generated it. There is a simple procedure to obtain the coordinates of this original cube from the coordinates of the new cube. For each coordinate $x_i$, we replace it by $x_i - 2m_i$, where $m_i$ is an integer chosen so that $0 \le x_i - 2m_i < 2$. We can now see that if two cubes agree in some coordinate, then the two cubes that generated them must agree in that coordinate. Similarly, suppose that two cubes differ by exactly 1 in some coordinate. Then $x_i - x'_i = 1$. Then the two corresponding cubes in the original have a difference in that coordinate of

$$x_i - 2m_i - (x'_i - 2m'_i) = x_i - x'_i - 2(m_i - m'_i) \equiv 1 \pmod 2$$

which since $0 \le x_i - 2m_i < 2$, must be either $\pm 1$. Thus, the original cubes in the fundamental domain must differ by exactly one in any coordinates in which the new cubes differ by exactly one. □

Before we continue, it might be worthwhile to give some example of period-2 tilings. One way to specify these tilings is to list the coordinates of the $2^n$ cubes which have coordinates at least 0 and less than 2. To check that this is a tiling, we need to check that any two cubes have one coordinate in which they differ by exactly 1. For it to be a counterexample to Keller's conjecture, any two cubes must differ in at least two coordinates. (If they only differ in one coordinate, then in this coordinate the difference must be 1 for the set to be a tiling, and so these two cubes will meet in a $n - 1$ dimensional face.)

In two dimensions, up to rotation and translation, all tilings must look like

$$\begin{array}{cc} 0 & 0 \\ 0 & 1 \\ 1 & a \\ 1 & a+1 \end{array}$$

This leads to two combinatorially different cases. Either $a = 0$, in which case the tiling is the planar grid, or $0 < a < 1$, in which case the tiling the planar grid with alternate pairs of columns shifted by $a$, and we have two pairs of cubes that each meet in an $1$ dimensional face.

In three dimensions, there is an interesting tiling which has properties no lattice tiling does. This tiling is

$$\begin{array}{ccc} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & a \\ 0 & 1 & a+1 \\ a & 0 & 1 \\ a+1 & 0 & 1 \\ 1 & a & 0 \\ 1 & a+1 & 0 \end{array}$$

One can check that if $0 < a < 1$, the two cubes with centers at $(0, 0, 0)$ and $(1, 1, 1)$ do not meet any other cubes in a 2-dimensional face. The other six cubes in the fundamental domain each meet one other cube in this domain in a 2-dimensional face. The combinatorial properties of this tiling, of course, do not depend on the value of $a$ as long as it is non-integral. For simplicity, we may as well take it to be $\frac{1}{2}$.

The next step in the reduction to the combinatorial problem that Corrádi and Szabó worked on was showing that if there is a counterexample to Keller's conjecture, then there is a counterexample with all coordinates either integral or half-integral. The examples we have just seen show the way to the first step in this argument. Let us concentrate on one dimension, let us say the first dimension, and consider the fractional part of the first coordinates of the cubes. Suppose the lower face of one of the cubes is at position $m_1 + a_1$, where $m_1$ is an integer and $0 \leq a_1 < 1$. Then the upper face of this cube is at position $m_1 + 1 + a_1$, and the lower face of all the cubes that meet this face also have position $m_1 + 1 + a_1$. Now, consider the set of cubes whose first coordinate has fractional part $a_1$. If we increase $a_1$ for all these cubes, this set will slide smoothly, without being blocked by any other cubes. Thus, we can replace the fractional part $a_1$ with any other fractional $a'_1$ with $0 \leq a'_1 < 1$. If $a'_1$ is not the fractional part of the first coordinate of any other cube, the combinatorial structure of the cube tiling will not change; so a counterexample to Keller's conjecture will remain a counterexample.

We thus see that for Keller's conjecture, the actual values of the fractional part of the coordinates in a cube tiling does not matter, but just the number of them. We will assume that one cube has its center at the origin, so the fractional value of $0$ will always be present in a cube tiling.. If there is just one other fractional value in some coordinate dimension, we can replace it by the value $\frac{1}{2}$, and obtain a combinatorially identical cube tiling. The question is: what to do if there is more than one fractional value in some dimension. In this case, what Szabó did was to increase the number of dimensions in the cube tiling while reducing the number of fractional values. We will again make this replacement one coordinate dimension at a time. We take one coordinate dimension, and for each value of that coordinate, we replace it with all of some set of values. If there are $l$ different values of the fractional part in this coordinate, we will need to replace this one dimension with $\lceil \log l \rceil$ dimensions. For example, if we have four possible values for the fractional part in some coordinate, say $0$, $a_1$, $a_2$, $a_3$, we use the following replacement table:

| value | replace by |
|---|---|
| $0$ | $00;\ 11$ |
| $1$ | $01;\ 10$ |
| $a_1$ | $0, \frac{1}{2};\ 1, \frac{3}{2}$ |
| $1 + a_1$ | $0, \frac{3}{2};\ 1, \frac{1}{2}$ |
| $a_2$ | $\frac{1}{2}, 0;\ \frac{3}{2}, 1$ |
| $1 + a_2$ | $\frac{1}{2}, 1;\ \frac{3}{2}, 0$ |
| $a_3$ | $\frac{1}{2}, \frac{1}{2};\ \frac{3}{2}, \frac{3}{2}$ |
| $1 + a_3$ | $\frac{1}{2}, \frac{3}{2};\ \frac{3}{2}, \frac{1}{2}$ |

One replaces each cube in the original $n$-dimensional tiling by two cubes in the $n+1$ dimensional tiling, as given by the above table. For example, the cube $(a_1, b_3, c_2)$ would be replaced by the two cubes $(0, \frac{1}{2}, b_3, c_2)$ and $(1, \frac{3}{2}, b_3, c_2)$. Thus, the $2^n$ cubes in the fundamental domain for $n$ dimensions are replaced by $2^{n+1}$ cubes in the fundamental domain for $n + 1$ dimensions. Recall that in a two-periodic tiling, the condition for it to be a tiling is that every pair of cubes in the fundamental domain have some coordinate dimension in which they differ by 1. One can check that this holds in the replacement tiling by checking that if two cubes differ by 1 in the coordinate being replaced, all pairs of their replacements differ by 1. This is sufficient because any cubes that differ by 1 in a coordinate not being replaced still differ by 1 in that coordinate after the replacement. For Keller's conjecture to be false in a two-periodic tiling, we need in addition that every pair of cubes in the fundamental domain differ in two coordinates. It is also easy to check that this property holds after the replacement if it held before. We need to check first that any two cubes in every set which we replace a single coordinate symbol by differ in at least two positions, and that no two replacement strings for different symbols are identical.

Generalizing the above replacement table to deal with $2^k$ different fractional values instead of $4$ is straightforward. We will replace the set $\{a_i, 1 + a_i\}$ with the $2^k$ sequences of $k$ coordinates which all have the same set of fractional values. There are $2^k$ possible sequences of length $k$ of the fractional values $0$, $\frac{1}{2}$, so we can assign one of these sequences to each $a_i$. We next need to split this set of $2^k$ sequences up into two sets such that no two sequences in the same set differ by 1 in exactly 1 place. There is only one way to do this, which is to split them up by the parity of the number of entries $\{0, \frac{1}{2}\}$ and the number of entries $\{1, \frac{3}{2}\}$.

We can now state Szabó's reformulation of Keller's conjecture is equivalent to the original conjecture.

**Conjecture 3 (Szabó)** *Every 2-periodic cube tiling of $\mathbb{R}^n$ where the cubes have integral and half-integral coordinates contains two cubes that meet in a full $n$-dimensional face.*

To check whether this reformulation holds in $n$ dimensions is a finite problem. (This would not prove Keller's original conjecture in $n$ dimensions, as the reduction procedure may increase the dimension.) There are only $4^n$ possible ways of putting cubes on integral or half-integral coordinates in the cube with sides 2 that forms the fundamental domain of our 2-periodic cube tiling. We only (!) need to check every set of $2^n$ of them to see whether it forms a valid tiling of $n$-dimensional space, and to see whether some pair of cubes in this tiling meet in an $n - 1$ dimensional face. Of course, as the dimension grows, this search becomes prohibitively expensive in terms of computation unless there is some more efficient way to go about it.

Corrádi and Szabó realized that this problem was a special case of a well-studied problem in computer science. This problem is the problem of finding maximum cliques, which comes from graph theory. A (combinatorial) graph is a set of vertices, together with some subset of unordered pairs of these vertices which are called edges. The maximum clique problem is to find the largest subset $S$ of vertices such that every pair of vertices $S$ have an edge between them. To turn the tiling problem into a clique problem, we make a vertex for each of the $4^n$ possible positions of a cube in the fundamental domain, and put a edge between any two of the vertices if they can be in the same counterexample to Keller's conjecture, i.e., if they differ by 1 in at least one coordinate (and so give rise to disjoint cubes), and they differ in at least two coordinates (and so do not give rise to two cubes sharing an $n - 1$ dimensional face). Now, if a clique of size $2^n$ exists in this graph with $4^n$ vertices, the 2-periodic tiling it generates must be a counterexample to Keller's conjecture. We can form a fundamental domain with the cubes corresponding to the $2^n$ vertices in the clique. If this domain is replicated with period 2, no two cubes can overlap, as otherwise they would both not have been contained in the clique. As the volume of these cubes is $2^n$, they must cover every point in $\mathbb{R}^n$ and so will generate a tiling. Thus, a clique in this graph of size $2^n$ would yield a counterexample to Keller's conjecture.

Unfortunately for the program of resolving Keller's conjecture by computer, the clique problem is one of a class of problems that computer scientists call NP-complete. What this means in practice is that no efficient algorithm is known for solving these problems, and most computer scientists believe that it is very unlikely that any such algorithm will be found. There are generally algorithms for solving these problems that are better than the first algorithm you might think of, but they are all still believed in the worst case to take time exponential in the size of the input. Corrádi and Szabó nevertheless did a computer search for counterexamples, and were able to show by computer that there were no counterexamples in dimension less than or equal to five. Note that this is one less than the number of dimensions which Perron was able to resolve by hand nearly fifty years earlier. Corrádi and Szabó did report the size of the largest cliques they found, and I reproduce these in the table below.

| dimension | clique size |
|---|---|
| 2 | 2 |
| 3 | 5 |
| 4 | 12 |
| 5 | 28 |
| 6 | 57 |

For dimensions 2 through 5, this is indeed the size of the largest clique in the graph. For dimension 6, their program was not only unable to rule out a clique of size 64, but also could not even find the maximum size clique—in dimension 6, the right answer is 60. In our research, Jeff Lagarias and I figured out that all the maximal cliques above belong to regular classes. In dimensions $d = 2$ through $4$, the maximal clique comes from a class that in general has $2^d - d$ elements. In dimensions $d = 4$ through $6$, the maximal clique comes from a class that has $2^d - 4$ elements. To see how hard this problem is, I later entered these Keller graphs as challenge graphs in a DIMACS challenge to design good clique-finding algorithms. While most of the entrants found the right answer in dimension 5, only one or two of the entrants found the right answer in dimension 6, and nobody was able to get close to the largest clique we know in dimension 7 (containing 124 vertices). Later, David Applegate (with my help) wrote a program that used some of the structure of this graph to perform an exhaustive search for a counterexample. While the program managed to show in a minute or two that there was not counterexample in dimension 6, it spent several days attacking the case of dimension 7 and made no discernible progress.

The effort of Corrádi and Szabó was not wasted, however. With Jeff Lagarias, I managed to combine the maximal clique that they found in 3 dimensions with the maximal clique they found in 4 dimensions to achieve a $3 \times 4 = 12$ dimensional counterexample to Keller's conjecture. This construction is detailed in the next section.

# 6 A Counterexample

The idea of the counterexample is the same as the idea that Szabó used to show that you only need to consider tilings with integral and half-integral coordinates when looking for a counterexample. We will again use a table to replace certain coordinate values with all of a set of vectors. Let us consider the 3-dimensional tiling given above, with $a = \frac{1}{2}$. The tiling was as follows:

$$\begin{array}{ccc} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} \\ \frac{1}{2} & 0 & 1 \\ \frac{3}{2} & 0 & 1 \\ 1 & \frac{1}{2} & 0 \\ 1 & \frac{3}{2} & 0 \end{array}$$

There are three pairs of cubes that share a face in this tiling. The idea we use is that if somehow we had a new coordinate $0'$ that was still at distance $1$ from the coordinate $1$, but was different from $0$, we could replace $0$ by $0'$ as follows in one of each of the pairs of cubes that share a 2-dimensional face to obtain a "counterexample" to Keller's conjecture.

$$\begin{array}{ccc} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & \frac{1}{2} \\ 0' & 1 & \frac{3}{2} \\ \frac{1}{2} & 0 & 1 \\ \frac{3}{2} & 0' & 1 \\ 1 & \frac{1}{2} & 0 \\ 1 & \frac{3}{2} & 0' \end{array}$$

In this "tiling," every pair of rows differs in two places, and every pair also has a difference of 1 in at least one coordinate. There is, of course, no number $0'$ that is both equal to and different from $0$. However, if we replace $0$, $0'$, and $1$ with a string of four coordinates as in the following table, we will effectively have simulated such a number, at the cost of multiplying the number of dimensions by 4. Here, for each coordinate that contains $0$, we replace this with all 12 possible entries in $S_0$, for each coordinate that contains $0'$ we replace it with all 12 entries in $S'_0$, and for each coordinate that contains 1, we replace it with all 4 entries in $S_1$ below. Thus, the cube $(0, 0, 0)$ is replaced by $12^3 = 1488$ cubes, and the cube $(1, 1, 1)$ is replaced by $4^3 = 64$ cubes.

| $S_0$ | $S'_0$ | $S_1$ |
|---|---|---|
| $0, 0, 0, 0$ | $0, B, 0, B$ | $0, 1, A, A$ |
| $0, 0, A, 1$ | $A, 0, A, A$ | $A, A, B, 1$ |
| $0, 1, A, B$ | $A, A, A, B$ | $1, B, 0, B$ |
| $0, 1, B, 0$ | $A, A, B, 0$ | $B, 0, 1, 0$ |
| $0, B, B, 1$ | $A, B, 1, B$ | |
| $A, 0, 1, 0$ | $A, B, B, A$ | |
| $1, A, 0, 0$ | $1, 1, A, A$ | |
| $1, A, A, 1$ | $B, 0, 0, A$ | |
| $1, 1, 1, 0$ | $B, 0, 1, 1$ | |
| $1, B, 0, A$ | $B, A, 0, B$ | |
| $1, B, 1, 1$ | $B, 1, 1, B$ | |
| $B, A, B, 1$ | $B, 1, B, A$ | |

where $A$ stands for $\frac{1}{2}$ and $B$ for $\frac{3}{2}$. The special properties of this replacement that makes it work is that $S_0 \cup S_1$ is the fundamental domain of a cube tiling, $S'_0 \cup S_1$ is the fundamental domain of a cube tiling. that there no two cubes that meet in a face within $S_0$, within $S'_0$, or within $S_1$, and that no cube appears twice in the union $S_0 \cup S'_0 \cup S_1$. That these sets have these properties is fairly easy to check. We still need to specify what to replace $\frac{1}{2}$ and $\frac{3}{2}$ by. It turns out that these replacements are much less constrained, and so are fairly easy to make. The choice we use in our paper is to add the vector $(\frac{1}{2}, 0, 0, 0)$ to each vector in the sets above to get $S_{1/2}$ and $S_{3/2}$. Specifically, $S_{1/2} = S_0 + (\frac{1}{2}, 0, 0, 0)$ and $S_{3/2} = S_1 + (\frac{1}{2}, 0, 0, 0)$

How did we find these sets? It turns out that they come from the maximum size clique in the graph found by Corrádi and Szabó corresponding to Keller's conjecture in five dimensions. This clique is of size 28, the size of $S_0 \cup S'_0 \cup S_1$. By deleting one coordinate (we have to choose the correct one), and sorting the other cubes into sets depending on the value of the deleted coordinate, we get the sets $S_0$, $S'_0$, and $S_1$. We were looking at the structure of the maximum cliques carefully in the hope that we could generalize them somehow to a counterexample to Keller's conjecture, and we were also looking for a construction of the general type that we ultimately found. Remarkably luckily, these two searches came together to produce a counterexample. This is not as much of a coincidence as it might at first appear. The clique of size 28 is obtained by making the replacement $0 \to S_0$, $0' \to S'_0$ and $1 \to S_1$ in the second coordinate starting with the following "clique" of size 3:

$$\begin{array}{cc} 0 & 0 \\ 1 & 0' \\ \frac{1}{2} & 1. \end{array}$$

In 2000, John Mackey found a counterexample in 8 dimensions. The construction for his counterexample uses similar techniques to ours, although they are somewhat more general. He starts with a four-dimensional cube tiling, and instead of using a table to replace single coordinates with sets, he uses a table to replace pairs of coordinates with sets of length four vectors, thus getting an 8-dimensional counterexample. The resulting cube tiling has a remarkable amount of symmetry, much more than any of our counterexamples; it has only three distinct classes of cubes and only two distinct classes of coordinates. I suspect this construction may have interesting structure that could be discovered with further study.

# 7 References

The first part of this lecture is drawn from the first chapter of the superb book by Sherman Stein and Sandor Szabó. The last chapter of this book contains a proof of Hajós' theorem, (or more precisely, a generalization of it by Rédei). Unfortunately, the MIT library seems to have lost the book. I do have my own copy available. Stein's survey article in the *American Mathematical Monthly* is also excellent. The earlier papers I refer to, many of which are in German, are cited in the more recent references.

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