1: A permutation is an *involution* if applying it twice gives the identity. Let us call $M_n$ the number of involutions in permutations of $n$ objects. Find a recurrence for $M_n$, and use it to obtain a differential equation for the generating function

$$g(x) = \sum_{k=0}^{\infty} \frac{M_k}{k!} x^k.$$ 

Note the $k!$ in the denominator. If we didn’t have it, $g(x)$ would not converge. To help you, the first few terms of the sequence $M_k$ are

$$M_0 = 1, \quad M_1 = 1, \quad M_2 = 2,$$
$$M_3 = 4, \quad M_4 = 10, \quad M_5 = 26.$$ 

You can count the number of involutions of $n$ objects by counting the number of partial pairings of the numbers $1 \ldots n$, where each number is either paired with one other or left a singleton. For example, we get $M_4$ by counting the following pairings

$$\{(1, 2)(3, 4), \quad (1, 3)(2, 4), \quad (1, 4)(2, 3), \quad (1, 2)(3, 4), \quad (1, 3)(2, 4),$$
$$\quad (1, 4)(2, 3), \quad (2, 3)(1, 4), \quad (2, 4)(1, 3), \quad (3, 4)(1, 2), \quad (1)(2)(3, 4)\}.$$ 

The actual answer, which should be a solution to the differential equation, is

$$g(x) = e^{x+x^2/2}.$$ 

2: Construct a spreadsheet (or program) that takes two different 7th degree polynomials and multiplies them together by taking the Fourier transform over the integers modulo the prime $p = 5393$, multiplying the resulting sequences, and taking the inverse Fourier transform. The integer 14 is a 16th root of unity modulo 5393. You will need to know the inverses of 14 and 16. These can be computed using the Euclidean algorithm, which we haven’t gone over in class, so I’m telling them to you. They are

$$14^{-1} = 14^{15} = 3467 \mod 5393$$
$$16^{-1} = 5056 \mod 5393$$

You can either use the method I sketched in class, which I will try to get lecture notes up on by tomorrow, or the method described in Prof. Kleitman’s lecture notes, which is slightly different (although the fundamental ideas are the same). I am using significantly larger prime than Prof. Kleitman uses in his homework assignment, which means that you can’t let the spreadsheet calculate $3467^{15}$ directly, or you will get integer overflow. You’ll have to make a row (or column) containing the numbers $3457^k \mod p$, and you can easily get the spreadsheet to calculate the $k$th of these from the $k-1$st.
3: Suppose you turn two integers $a$ and $b$ into polynomials by replacing $10^k$ in their decimal expansions with $x^k$, multiplying the two polynomials using your spreadsheet above, and you get the polynomial

$$54x^8 + 120x^7 + 151x^6 + 165x^5 + 224x^4 + 183x^3 + 114x^2 + 73x + 63$$

Turn this back into the integer $ab$.

4: Show that if you have a 16th root of unity $r \neq 1$ modulo some prime $p$, then

$$1 + r + r^2 + r^3 + r^4 + \ldots + r^{15} = 0 \mod p.$$ 

This is a crucial fact (and is true over all fields) in terms of making the Fourier transform over a field work properly.