Properties of an EPR pair

If you measure the two halves of an EPR pair using the same basis, you always get orthogonal measurement outcomes.

$$\begin{aligned} |0\rangle|1\rangle - |1\rangle|0\rangle &= (\alpha|0\rangle - \beta|1\rangle)(\beta^*|0\rangle + \alpha^*|1\rangle) \\ &\quad -(\beta^*|0\rangle + \alpha^*|1\rangle)(\alpha|0\rangle - \beta|1\rangle) \end{aligned}$$

Physical explanation for spin $\frac{1}{2}$ particles:
$\frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle)$ has spin 0, no matter what basis it's expressed in.

Does one of the qubits "know" what basis it should be in after you measure the other one?

Entanglement:

Einstein, Podolsky, Rosen (1935)
Quantum mechanics is a very strange theory, in that it doesn't allow for local descriptions of local reality, and thus it must be incomplete.

Schrödinger (1935)
Motivated by EPR, he introduced the term *entanglement (verschränkung)*.

Bell (1964)
Bell's inequalities — completing quantum mechanics won't make the strangeness go away; probability distributions arising in quantum mechanics can't be explained classically without some kind of nonlocality.

Greenberger, Horne, Shimony, Zeilinger (1990)
"Bell's Theorem without Inequalities."

GHZ state (Greenberger, Horne, Zeilinger)

$$\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

In the GHSZ experiment, we measure each qubit either in the basis $\frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$ or the basis $\frac{1}{\sqrt{2}}(|0\rangle \pm i\,|1\rangle)$.

What do we get?

Let

$$
\begin{aligned}
|+\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \\
|-\rangle &= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).
\end{aligned}
$$

The state

$$
\begin{aligned}
|\phi_{\mathrm{GHZ}}\rangle &= \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \\
&= \frac{1}{2}(|+++\rangle + |+--\rangle \\
&\quad + |-+-\rangle + |--+\rangle\,)
\end{aligned}
$$

Thus, if you measure $|\phi_{\mathrm{GHZ}}\rangle$ in the $\{|+\rangle, |-\rangle\}$ basis, you always get an even number of $|-\rangle$'s.

(Since $|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$ and $|1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$, coefficients of terms with an odd number of $|-\rangle$s cancel in the expansion).

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \qquad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle),$$

$$\left|C^+\right\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle), \qquad \left|C^-\right\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle).$$

If you measure $|\phi_{\mathrm{GHZ}}\rangle$ using the $|C^{\pm}\rangle$ basis for two of the qubits, and using the $|\pm\rangle$ basis for the other qubit, you always get an *odd* total number of $|-\rangle \cup |C^-\rangle$s.

$$\begin{aligned}
|\phi_{\mathrm{GHZ}}\rangle &= \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \\
&= \frac{1}{2}\Big(\left|+C^+C^-\right\rangle + \left|+C^-C^+\right\rangle \\
&\quad + \left|-C^+C^+\right\rangle + \left|-C^-C^-\right\rangle\Big)
\end{aligned}$$

(Almost same calculation as previous slide; but an odd total because $i \cdot i = -1$.)

Now, we consider the results of measuring $|\phi_{\mathrm{GHZ}}\rangle$ using the four bases given in the table below, keeping track of the parity of the number of $\{|-\rangle, |C^-\rangle\}$ in the outcome. Here, D means the basis $\{|\pm\rangle\}$ and C the basis $\{|C^\pm\rangle\}$.

qubit 1	qubit 2	qubit 3	parity of $\{-, C^-\}$
D	D	D	0
D	C	C	1
C	D	C	1
C	C	D	1

No deterministic set of outcomes that specify the outcomes of each measurement for every qubit can give this, since the total parity of the outcomes in each of the first three columns would have to be 0.

Thus, any local realistic theory must disagree with quantum mechanics for at least one quarter of its predictions.

Applications of entanglement include

- quantum teleportation
- superdense coding
- quantum cryptography
- quantum computing

When are two quantum systems *entangled*?

Given a joint density matrix ρ_{AB}, we could ask

- Can we create ρ_{AB} in two quantum laboratories using only a telephone line to communicate between them?
- Can we make measurements whose outcomes do not satisfy Bell's inequality or a generalization (e.g., CHSH) of it?
- Can we make EPR pairs from many copies of ρ_{AB} (i.e., $\rho_{AB}^{\otimes n}$)?
- Can we use a classical channel from Alice to Bob and a supply of shared states ρ_{AB} to teleport quantum information?

ρ_{AB} is called *separable* if it can be created in two quantum laboratories which only use a telephone line to communicate.

A separable state can be made by Alice (in one laboratory) flipping a many-sided coin, sending the results to Bob, and then both Alice and Bob making a specified pure state depending on the outcome.

That is, a separable state can be expressed as

$$\rho_{AB} = \sum_{i=1}^{k} p_i |v_i\rangle\langle v_i| \otimes |w_i\rangle\langle w_i|.$$

Any state which is not separable is called entangled.

Consider a pure state Ψ_{AB} shared between Alice and Bob.

Theorem (Schmidt decomposition)
Let $\lambda_i, |v_i\rangle$ be the eigenvectors and eigenvalues of $\mathrm{Tr}_B|\Psi_{AB}\rangle\langle\Psi_{AB}|$. Then

$$\Psi_{AB} = \sum_i \sqrt{\lambda_i}|v_i\rangle|w_i\rangle$$

where

$$\langle v_i|v_j\rangle = \delta_{ij}$$

$$\langle w_i|w_j\rangle = \delta_{ij}$$

Thus, the $|w_i\rangle$ are the eigenvectors of the matrix $\mathrm{Tr}_A|\Psi_{AB}\rangle\langle\Psi_{AB}|$, and

$$H(\mathrm{Tr}_A|\Psi_{AB}\rangle\langle\Psi_{AB}|) = H(\mathrm{Tr}_B|\Psi_{AB}\rangle\langle\Psi_{AB}|).$$

If ρ_{AB} is a pure state, then we have a good definition of entanglement.

$$E_P(\rho_{AB}) = H(\mathrm{Tr}_A \rho_{AB}) = H(\mathrm{Tr}_B \rho_{AB})$$

Theorem (Bennett, Bernstein, Popescu, Schumacher)

- $nE_P(\rho_{AB}) - o(n)$ EPR pairs can be obtained from n copies of ρ_{AB}, using local quantum operations, with high probability.
- n copies of a state very close to ρ_{AB} can be obtained from $nE_P(\rho_{AB}) + o(n)$ EPR pairs, using local quantum operations and classical communication.

To prove this theorem, we will use Nielsen's non-asymptotic characterization of when one pure entangled states can be created from another pure entangled state.

Theorem (Nielsen)
If we have a pure entangled state ρ_{AB} and a pure entangled state σ_{AB}, both shared between Alice and Bob, then using local operations on Alice's and Bob's states, we can create ρ_{AB} from σ_{AB} if and only the eigenvalues λ_σ of $\mathrm{Tr}_B\sigma$ are majorized by the eigenvalues λ_ρ of $\mathrm{Tr}_B\rho$.

Definition: λ_ρ majorizes λ_σ if and only if

$$\sum_{i=1}^{k} \lambda_{\rho,i} \geq \sum_{i=1}^{k} \lambda_{\sigma,i}$$

when the eigenvalues are given in decreasing order.

Proof (one direction) for Alice and Bob both holding qubits.

Let us take without loss of generality

$$\psi_\rho = \alpha|00\rangle + \beta|11\rangle$$

where $\alpha \geq \beta$, $\alpha^2 + \beta^2 = 1$. The eigenvalues of $\mathrm{Tr}_B \rho$ are α^2 and β^2.

Alice can measure her state with the quantum instrument derived from matrices $\frac{1}{\sqrt{2}} A_+$ and $\frac{1}{\sqrt{2}} A_-$, where

$$A_+ = \begin{pmatrix} x & y \\ y & x \end{pmatrix} \qquad A_- = \begin{pmatrix} x & -y \\ -y & x \end{pmatrix}$$

where $x^2 + y^2 = 1$. This is a valid instrument since

$$\frac{1}{2}\left(A_+^\dagger A_+ + A_-^\dagger A_-\right) = x^2 + y^2 = 1.$$

$$\text{Recall} \quad \psi_\rho = \alpha|00\rangle + \beta|11\rangle$$

$$A_+ = \begin{pmatrix} x & y \\ y & x \end{pmatrix} \qquad A_- = \begin{pmatrix} x & -y \\ -y & x \end{pmatrix}$$

After Alice's measurement (suppose she obtains A_+), the state is

$$|\Psi_+\rangle = (A_+ \otimes I)\begin{pmatrix} \alpha \\ 0 \\ 0 \\ \beta \end{pmatrix} = x\alpha|00\rangle + y\alpha|10\rangle + y\beta|01\rangle + x\beta|11\rangle.$$

$$\mathrm{Tr}_B|\Psi_+\rangle\langle\Psi_+| = \begin{pmatrix} \alpha^2x^2 + \beta^2y^2 & xy \\ xy & \alpha^2y^2 + \beta^2x^2 \end{pmatrix}$$

This has eigenvalues $\frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 4\alpha^2\beta^2(x^2 - y^2)}$. These range between (α^2, β^2) $(x = y = 1/\sqrt{2})$ and (1,0) $(x = 1, y = 0)$.

Thus, if Alice and Bob want to go from ψ_ρ to ψ_σ, where the eigenvaules of $\mathrm{Tr}_B\sigma$ majorize $\mathrm{Tr}_B\rho$, she can do it when they have qubits.

Now, suppose they have higher dimensional states. They can alter two of the eigenvalues at a time by using the quantum instrument with matrices

$$\begin{pmatrix} A_+ & 0 \\ 0 & I_{d-2} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A_- & 0 \\ 0 & I_{d-2} \end{pmatrix}$$

By repeatedly using such quantum instruments, we can go from any ρ to σ if the eigenvalues of σ majorize those of ρ.

Suppose we have many copies of a pure state $|\psi\rangle^{\otimes m}$, and we want to make $|\phi\rangle^{\otimes n}$, where

$$|\phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

Expanding, we get

$$\begin{aligned} |\phi\rangle &= \frac{1}{2^{n/2}}(|00\rangle + |11\rangle)^{\otimes n} \\ &= \frac{1}{2^{n/2}} \sum_{i=0}^{2^n-1} |i_A\rangle \otimes |i_B\rangle \end{aligned}$$

This is an equal superposition of 2^n vectors $|i_A\rangle \otimes |i_B\rangle$ where the $|i_A\rangle$ are orthogonal vectors in Alice's subsystem, and the $|i_B\rangle$ are orthogonal vectors in Bob's subsystem.

This is called a *maximally entangled state* in 2^n dimensions.

Asyptotic pure state entanglement

Let

$$|\psi\rangle = \sum_{i=1}^{d} \sqrt{\lambda_i} |v_i\rangle |w_i\rangle$$

Then

$$|\psi\rangle^{\otimes n} = \sum_{n=k_1+\ldots+k_d} \sqrt{\lambda_1^{k_1} \lambda_2^{k_2} \ldots \lambda_d^{k_d}} \sum_{I \in \binom{n}{k_1 \ldots k_d}} |v_I\rangle \otimes |w_I\rangle$$

Nearly all the weight is in eigenvectors with eigenvalues close to $2^{-nH(\lambda_1,\lambda_2,\ldots,\lambda_d})$. If we take the state very close to this which drops all other eigenvalues, this is majorized by the state consisting of $nH(\lambda_1, \lambda_2, \ldots, \lambda_d) - o(n)$ EPR pairs, and majorizes the state consisting of $nH(\lambda_1, \lambda_2, \ldots, \lambda_d) + o(n)$ EPR pairs,

Thus LOCC operations can asymptotically convert any pure state with E entanglement to any other pure state with E entanglement.

How much communication do you need to create a state $|\phi\rangle^{\otimes n}$ starting from around $nE(\phi)$ EPR pairs? It turns out you need $O(\sqrt{n})$ bits of classical communication (Upper bound: Lo and Popsecu; Lower bound: Harrow and Lo).

How to create EPR pairs from n copies of $|\phi\rangle$ using local operations but without any communication.
We have state

$$|\psi\rangle^{\otimes n} = \sum_{n=k_1+\ldots+k_d} \sqrt{\lambda_1^{k_1}\lambda_2^{k_2}\ldots\lambda_d^{k_d}} \sum_{I\in\binom{n}{k_1\ldots k_d}} |v_I\rangle \otimes |w_I\rangle$$

Measure the number of times each eigenvector appears, i.e., the values $k_1 \ldots k_d$.
This projection measurement gives us a maximally entangled state in $\binom{n}{k_1\ldots k_d}$ dimensions. Can obtain $\log_2 \binom{n}{k_1\ldots k_d}$ EPR pairs from this.
With high probability,

$$\log_2 \binom{n}{k_1 \ldots k_d} \approx H(\lambda_1, \ldots \lambda_d)n.$$

What about measuring entanglement of mixed states?

Entanglement cost: $E_C(\rho)$

$$\lim_{n\to\infty} \frac{1}{n} \text{(Number EPR pairs needed to make approximation of } \rho^{\otimes n}.\text{)}$$

Distillable entanglement: $E_D(\rho)$

$$\lim_{n\to\infty} \frac{1}{n} \text{(Number approximate EPR pairs obtainable from } \rho^{\otimes n}. \qquad \text{)}$$

The protocols above may use only local operations and classical communication.

Clearly, $E_C \geq E_D$. For mixed states, we can have $E_C > E_D$.

Conjectures:

(1) Entanglement cost, E_C, is additive:

$$E_F(\rho_1 \otimes \rho_2) = E_F(\rho_1) + E_F(\rho_2).$$

(2) E_D is not additive
(A conjectured counterexample exists)

(3) Entanglement of formation, E_F, can be thought of as one-shot entanglement cost E_F, where

$$E_F = \min_{\rho = \sum_i p_i \rho_i} \sum_i p_i E_P(\rho_i)$$

and the ρ_i are pure states (rank 1).

Another Conjecture

It turns out that conjectures (1) and (3) on the previous slide would follow from a proof of additivity of entanglement of formation, E_F.

$$E_F(\rho_1 \otimes \rho_2) = E_F(\rho_1) + E_F(\rho_2).$$

where

$$E_F = \min_{\rho = \sum_i p_i \rho_i} \sum_i p_i E_P(\rho_i)$$

and the ρ_i are pure states (rank 1).

This follows from the theorem

$$E_C(\rho) = \lim_{n \to \infty} \frac{1}{n} E_F(\rho^{\otimes n})$$

There is a characterization of entangled states in 2×2 and 2×3 dimensions.

Partial transpose: take transpose of Bob's half.

What happens for $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$?

$$\rho = \left(\begin{array}{cc|cc} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{array}\right)$$

Take the transpose of Bob's part:

$$\rho^{PT} = \left(\begin{array}{cc|cc} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ \hline 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{array}\right)$$

Eigenvalues: $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}$.

If ρ is not entangled, the partial transpose has positive eigenvalues.

Proof:

$$\rho = \sum_i p_i |v_i\rangle\langle v_i| \otimes |w_i\rangle\langle w_i|$$

Then

$$\rho^{PT} = \sum_i p_i |v_i\rangle\langle v_i| \otimes |w_i^*\rangle\langle w_i^*|$$

Theorem (Peres, Horodecki^3):

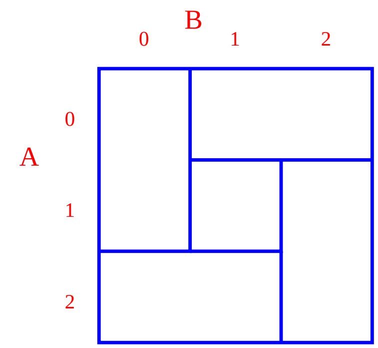
If $\rho^{PT} \geq 0$, then $E_D = 0$: no distallable entanglement.
Proof idea: LOCC operations cannot create negative eigenvalues

If $\rho \in \mathbb{C}^2 \otimes \mathbb{C}^3$, $\rho^{PT} \geq 0$ if and only if

$$\rho = \sum p_i |v_i\rangle\langle v_i| \otimes |w_i\rangle\langle w_i|,$$

i.e., ρ is not entangled.

not true if ρ in any larger entangled system.



There are *bound entangled* states, so that $\rho^{PT} \geq 0$, implying $E_D = 0$, but so ρ cannot be created without using entanglement.

$$
\begin{aligned}
|v_1\rangle &= \tfrac{1}{3}\left(|0\rangle - |1\rangle + |2\rangle\right) \otimes \left(|0\rangle - |1\rangle + |2\rangle\right) \\
|v_2\rangle &= \tfrac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \otimes \left(|0\rangle\right) \\
|v_3\rangle &= \tfrac{1}{\sqrt{2}}\left(|2\rangle\right) \otimes \left(|0\rangle + |1\rangle\right) \\
|v_4\rangle &= \tfrac{1}{\sqrt{2}}\left(|0\rangle\right) \otimes \left(|1\rangle + |2\rangle\right) \\
|v_5\rangle &= \tfrac{1}{\sqrt{2}}\left(|1\rangle + |2\rangle\right) \otimes \left(|2\rangle\right)
\end{aligned}
$$

$|v_i\rangle$ and $|v_{i+1}\rangle$ orthogonal in first coordinate.
$|v_i\rangle$ and $|v_{i+2}\rangle$ orthogonal in second.
No product state is orthogonal to all five vectors (it would have to be orthogonal to three vectors in one of the coordinates).

Bound Entanglement

Consider the state

$$\rho = \frac{1}{4} \left(I - |v_1\rangle\langle v_1| - |v_2\rangle\langle v_2| - |v_3\rangle\langle v_3| - |v_4\rangle\langle v_4| - |v_5\rangle\langle v_5| \right)$$

Since $I^{PT} = I$, and $(|v_i\rangle\langle v_i|)^{PT} = (|v_i\rangle\langle v_i|)$, we have that $\rho^{PT} = \rho$ is positive, so $E_D(\rho) = 0$. Thus no EPR pairs are distillable from it.

support(ρ) has no tensor product vectors in it, so ρ is not separable. Thus, ρ cannot be created without entanglement.

In fact, one can show that $E_F(\rho) > 0$, so that a linear number of EPR pairs are required to create ρ.

Theorem (Yang, Horodecki, Horodecki, Synak-Radtke):
All non-separable states have $E_C > 0$.
The proof takes several steps. The key is a definition called the classical correlation of two quantum states.

$$\begin{aligned}
C_{\rightarrow}(\rho_{AB}) &= \max_{A_i^\dagger A_i} H(\rho_B) - \sum_i p_i H(\rho_B^i) \\
C_{\leftarrow}(\rho_{AB}) &= \max_{A_i^\dagger A_i} H(\rho_B) - \sum_i p_i H(\rho_B^i) \\
C_{\leftrightarrow}(\rho_{AB}) &= \max\{C_{\rightarrow}(\rho_{AB}), C_{\leftarrow}(\rho_{AB})\}
\end{aligned}$$

Intuitively, $C_{\leftarrow}$ is how much classical information Alice can learn about Bob's state by performing a measurement on her state.
$C_{\leftarrow}(\rho_{AB}) = 0$ if and only if $\rho_{AB} = \rho_A \otimes \rho_B$.

We now define

$$\begin{aligned} G_{\rightarrow}(\rho_{AB}) &= \inf \sum_i p_i C_{\rightarrow}(\rho^i_{AB}) \\ G_{\leftarrow}(\rho_{AB}) &= \inf \sum_i p_i C_{\leftarrow}(\rho^i_{AB}) \qquad \text{where} \qquad \sum_i p_i \rho^i_{AB} = \rho_{AB} \\ G_{\leftrightarrow}(\rho_{AB}) &= \inf \sum_i p_i C_{\leftrightarrow}(\rho^i_{AB}) \end{aligned}$$

Theorem: $G = 0$ if and only if ρ_{AB} is separable.
Proof: Follows from $C = 0$ if and only if ρ_{AB} is a tensor product, continuity of G, and a compactness argument.

Duality:

Let $|\phi\rangle_{ABC}$ be a pure state.

$$H(\rho_A) = E_F(\rho_{AC}) + C_{\leftarrow}(\rho_{AB})$$

This holds because $C_{\leftarrow}$ is a minimization over all measurements of ρ_{AB} on Bob's half. These measurements have an infimum where Bob's measurements is a POVM with just rank one matrices. This induces a decomposition of ρ_{AC} into pure states, which is the same decomposition of entanglement of formation.

Lemma: For any fourpartite pure state $|\phi\rangle_{AA'BB'}$, we have

$$E_F(\phi_{AA':BB'}) \geq E_F(\rho_{A:B}) + C_{\leftrightarrow}(\rho_{A':B'})$$

Proof: Use duality

$$\begin{aligned} E_F(\phi_{AA':BB'}) = H(\rho_{AB}) &= E_F(\rho_{AA':B}) + C_{\leftarrow}(\rho_{AA':B'}) \\ &\geq E_F(\rho_{A:B}) + C_{\leftarrow}(\rho_{A':B'}) \end{aligned}$$

and a similar argument gives

$$E_F(\phi_{AA':BB'}) = H(\rho_{A'B'}) \geq E_F(\rho_{A:B}) + C_{\rightarrow}(\rho_{A':B'})$$

Theorem: For a four-partite state $\rho_{AA'BB'}$,

$$E_F(\rho_{AA':BB'}) \geq E_F(\rho_{A:B}) + G_{\leftrightarrow}(\rho_{A':B'}).$$

Proof:

$$\begin{aligned} E_F(\rho_{AA':BB'}) &= \sum_i p_i H(\rho^i_{AA'}) \\ &\geq \sum_i p_i E_f(\rho^i_{A:B}) + \sum_i p_i C_{\leftrightarrow}(\rho^i_{A'B'}) \\ &\geq E_F(\rho_{A:B}) + G_{\leftrightarrow}(\rho_{A':B'}) \end{aligned}$$

Theorem: For a four-partite state $\rho_{AA'BB'}$,

$$E_F(\rho_{AA':BB'}) \geq E_F(\rho_{A:B}) + G_{\leftrightarrow}(\rho_{A':B'}).$$

Corollary:

$$\begin{aligned} nE_C(\rho) &\geq E_F(\rho^{\otimes n}) \\ &\geq E_F(\rho^{\otimes n-1}) + G_{\leftrightarrow}(\rho) \\ &\geq E_F(\rho^{\otimes n-2}) + 2G_{\leftrightarrow}(\rho) \\ &\geq E_F(\rho) + (n-1)G_{\leftrightarrow}(\rho). \end{aligned}$$

So this gives

$$E_C(\rho) \geq G_{\leftrightarrow}(\rho) \geq 0.$$

Other Measures Useful for Entanglement

- Relative entropy of entanglement

$$\inf_{\sigma \in D} \mathrm{Tr}\rho(\log \rho - \log \sigma)$$

  where D is the set of separable matrices.

- Logarithmic negativity

$$\log ||\rho^{PT}||_1 = \log(1 + 2N(\rho))$$

  where $N(\rho)$ is the sum of the negative eigenvalues.

Relative entropy of entanglement is between $E_D(\rho)$ and $E_F(\rho)$, and is not additive.

Logarithmic negativity is an upper bound on $E_D(\rho)$, and is additive.

Even more measures

- Bounds from positive partial transpose distillable entanglement (Rains), (Audenaert, de Moor, Vollbrecht, Werner)
- Squashed entanglement (Christandl and Winter)

$$E_{sq}(\rho_{AB}) = \inf \left\{ \tfrac{1}{2} I(A;B|E) \text{ where } \rho_{ABE} \text{ is an extension of } \rho_{AB} \right\}$$

$$I(A;B|E) = H(\rho_{AE}) + H(\rho_{BE}) - H(\rho_{ABE}) - H(\rho_E)$$

The Rains bound is computatable via a semi-definite program, and always larger than distillable entanglement.
It is related to bounds by Audenaert et al.
Squashed entanglement is additive, and $E_C \geq E_{sq} \geq E_D$.