Homework 18.095, Keller's Conjecture, Prof. Peter Shor,

1. The order of a group element is the minimum integer $r > 0$ such that $ra = \bar{0}$, where $\bar{0}$ is the identity element of the group. (As in the lecture, we use $+$ for the group operation; ra is the element obtained by adding a to $\bar{0}$ r times.)

1a. Let L be the lattice tiling by unit cubes with basis vectors $(1, 0)$ and $(\frac{1}{2}, 1)$, and L' be the lattice tiling with basis vectors $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$. Consider the factor group $G = L'/L$. How many elements does it contain? Does it have an element of order 4?

1b. Answer the same questions as in 1a for the tiling L with basis vectors $(1, 0)$ and $(0, 1)$, and the tiling L' with basis vectors $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$.

2. Complete the following proof that Keller's conjecture is true in 3 dimensions.

Note: Keller's conjecture in k dimensions is the statement that every tiling of $\mathbb{R}^k$ by unit cubes contains two cubes that meet in a a $k - 1$ dimensional face.

Let us consider a counterexample. By results from the lecture, we may assume w.l.o.g. that this counterexample is of period 2. We may assume that one of the cubes has a center at $(0, 0, 0)$. Now, there must be a cube with a center (a, b, c) with $1 \le a < 2$ and $0 \le b, c < 1$. In order for it not to overlap the first cube, we must have $a = 1$. If b, c are both 0, then we have two cubes meeting in a 2-dimensional face, and so the tiling is not a counterexample. Thus, there are two possibilities: either one of b, c is 0, or neither of b, c is 0.

We now show we can assume that one of b, c is 0. Assume that both b, c are not 0. By the same argument that led to the cube with center $(1, b, c)$, we have that there is also a cube with center $(d, 1, e)$, with $0 \le d, e < 1$. Thus, we have a partial cube tiling

$$\begin{array}{ccc} 0 & 0 & 0 \\ 1 & b & c \\ d & 1 & e \end{array}$$

Now, since $b > 0$, in order for the second and third cubes not to overlap we must have either $|1 - d| = 1$ or $|c - e| = 1$. The second possibility is impossible since $0 \le c, e < 1$, so we deduce that $d = 0$. But now, we can interchange the first and second coordinates to obtain two cubes with

$$\begin{array}{ccc} 0 & 0 & 0 \\ 1 & 0 & e \end{array}$$

We now consider a third cube, with center f, g, h where $1 \le f, h < 2$ and $0 \le g < 1$. For this cube to be disjoint from the cube $(1, 0, e)$, we must have $h = e + 1$. Now, for this to be disjoint from the cube with center $(2, 0, 2)$, we must have that at least one of f and h is 1. This must be f, since $h = e + 1$ and $e \ne 0$. Thus, this third cube has center $(1, g, e + 1)$, and we have a partial cube tiling

$$\begin{array}{ccc} 0 & 0 & 0 \\ 1 & 0 & e \\ 1 & g & e+1 \end{array}$$

with $0 < e < 1$, $0 \le g < 1$. But now consider that if $g = 0$, we have two cubes which meet in a 2-dimensional face, so we must have $0 < g < 1$.