18.424 Homework # 3

Due Monday, March 19

Do exercises 4.2 and 7.12.

1: Suppose that we have a channel which flips a biased coin, and with probability p sends the input through one channel A and with probability $q = 1 - p$ sends it through a different channel B. Suppose further that A and B have distinct output alphabets; so that the receiver can tell whether the input went through channel A or B by looking at the output.
1a: Compute the capacity of this channel. This capacity should be expresed as the maximum over an input distribution X of an expression involving channels A and B.
1b: Use the expression from 1a to solve exercise 7.13 in Cover and Thomas.

2: The Burrows Wheeler transform
Suppose we have a stationary, aperiodic, irreducible, stochastic process where the n'th state is dependent only on the previous k states. That is,

$$P(X_n|X_{n-1},\ldots,X_1) = P(X_n|X_{n-1},\ldots,X_{n-k}) = P(X_{k+1}|X_k,\ldots,X_1).$$

Suppose further that the alphabet is of size m, so each X_i can take on m different values. Consider the Burrows-Wheeler transform of the reverse of the output, $X_n, X_{n-1}, \ldots, X_1$.
2a: Show that the output of the BWT can be divided into at most m^k different consecutive subsequences where the expected number of of symbol s_i in each subsequence is the expected length of that subsequence times

$$P(X_{k+1} = s_i | X_k = s_{r_k}, \ldots, X_1 = s_{r_1})$$

where the values of $s_{r_k}, \ldots, s_{r_1}$, depend on the subsequence.
The irreducibility of the Markov chain together with the law of large numbers means that for any ϵ, as $n \to \infty$, each subsequence is with high probability within $1 \pm \epsilon$ of its expected length.
2b: Give an estimate for the entropy of the output of the Burrows Wheeler transform using 2a. Consider the expected length of each of the subsequences, as well the positions of the divisions between them. Compare it with the bound derived from (4.77) in chapter 4 of Cover and Thomas. Use this comparison to conclude that as $n \to \infty$, each of the subsequences has nearly the same entropy as an i.i.d. sequence with the same probability distribution on the symbols.

3: Channels with memory.
Suppose we have a binary channel with an internal state Z_i, which is either 0 or 1. Suppose it obeys the dynamics

$$\begin{aligned} Y_i &= X_i + Z_i \pmod 2 \\ Z_{i+1} &= Z_i \quad \text{with probability } 1-p \\ Z_{i+1} &= 1 - Z_i \quad \text{with probability } p \end{aligned}$$

3a: Compute upper and lower bounds of the capacity of the channel by considering blocks of 2 inputs at a time, $X_{2i-1}X_{2i}$ (i.e. the second extension of the channel). For the lower bound, let the receiver assume that Z_i is equally probably to be 0 or 1 for $i = 1, 3, 5, \ldots$. For the upper bound, assume the receiver knows the value of Z_i at these times.

3b: Use the same procedure for blocks of length k to obtain better upper and lower bounds on the capacity of the channel.