Problem 1-1.

Part 1: Prove: if there is a p, $0 \le p \le 1$ such that $\binom{n}{k}p^{\binom{k}{2}} + \binom{n}{t}(1-p)^{\binom{t}{2}} \le 1$ then the Ramsey number $r(k,t)$ satisfies $r(k,t) > n$.

Suppose we take the complete graph K_n and color its vertices red with probability p and blue with probability $1-p$. The probability that any particular set of k vertices forms a red clique is $p^{\binom{k}{2}}$, and there are $\binom{n}{k}$ sets of red vertices, so the probability that there is a red clique of size k is at most $\binom{n}{k}p^{\binom{k}{2}}$. Similarly, the probability that there is a blue clique of size t is at most $\binom{n}{t}(1-p)^{\binom{t}{2}}$. If the sum of these two probabilities is less than 1, then there is a positive probability that there is no red clique of size k and no blue clique of size t, showing that $r(k,t) > n$.

Part 2: Show that $r(4,t) \ge \Omega(t^{3/2}/(\log t^{3/2}))$.

We deal with the first term first. This term is $\binom{n}{4}p^6 < n^4p^6/24$. Thus, if we take $p = n^{-2/3}$, we are guarnteed that the first term is at most $1/24$. Now, the second term is, after substituting in our expression for p,

$$\binom{n}{t}(1 - n^{-2/3})^{\binom{t}{2}}$$

We use $\binom{n}{t} \le (ne/t)^t$ and $(1 - n^{-2/3}) \le \exp(n^{-2/3})$ to bound this term by

$$\exp(t\log(ne/t) - n^{-2/3}t(t-1)/2) = \exp(t(\log n) - tn^{-2/3}(t-1)/2 - t\log(t/e))$$

which we would like to show is less than $23/24$. The third term in the exponential is negative and relatively small, so we ignore it. Setting the sum of the first two terms negative, we get

$$\log n < n^{-2/3}(t-1)/2$$

which can be seen to be satisfied for large n if $n = (t/\log t)^{3/2}$.

Problem 1-8 Let $N = \max N_i$. Suppose we pick a binary string s of length N at random. The probability that a string f of length i is a prefix of s is 2^{-i}. If the set F of strings is unique-prefix decodable, then no two strings in F can simultaneously be a prefix of s, so letting $|f|$ be the length of f

$$\sum_{f \in F} 2^{-|f|} \le 1,$$

or

$$\sum_i N_i 2^{-i} \le 1.$$

Problem 1 - 10: Let A be an $n \times n$ matrix with pairwise distinct entries. Then there is a permutation of the rows of A so that no column in the permuted matrix contains an increasing subsequence of length at least $c\sqrt{n}$.

Let us consider a random permutation. We bound the probability that there is an increasing subsequence of $c\sqrt{n}$. The probability that any particular subsequence of length $c\sqrt{n}$ is increasing is $1/(c\sqrt{n})!$ and there are $\binom{n}{c\sqrt{n}}$ such subsequences. We thus have an upper bound of

$$\binom{n}{c\sqrt{n}} \frac{1}{(c\sqrt{n})!}$$

on the probability that there is an increasing subsequence of length $c\sqrt{n}$. If this probability is less than $1/n$, then by linearity of expectation, the expected number of rows that have such an increasing subsequence is less than 1, so there is a permutation such that no row has a long increasing subsequence. Using Stirling's formula $n! \approx (n/e)^n (2\pi n)^{1/2}$, we see that

$$\binom{n}{c\sqrt{n}} \frac{1}{(c\sqrt{n})!} \approx \gamma n^{-3/4} \frac{n^n}{(n - c\sqrt{n})^{n - c\sqrt{n}} (c\sqrt{n})^{c\sqrt{n}}} \frac{e^{c\sqrt{n}}}{(c\sqrt{n})^{c\sqrt{n}}}$$

for some constant γ. Dropping the $\gamma n^{-3/4}$ term (which will be dwarfed by the exponential terms) and simplifying, we get

$$\left(1 - \frac{c}{\sqrt{n}}\right)^{-(n - c\sqrt{n})} \left(\frac{ne}{c^2 n}\right)^{c\sqrt{n}} \leq \left(\frac{e}{c}\right)^{c\sqrt{n}},$$

showing that choosing $c > e$ gives the result.

Problem 2-2 Show that for any set A of n nonzero reals numbers, there is a subset $B \subset A$ such that $|B| \geq cn$ and for $b_i \in B$, it is not true that

$$b_1 + 2b_2 = 2b_3 + 2b_4$$

First, let's consider arithmetic modulo 1 on the reals, so $a + b \pmod 1$ is the fractional part of $a + b$. I claim that if $B \subseteq (1/2, 2/3) \pmod 1$, then no four elements of B can satisfy this equation. This holds because $3 \cdot \frac{1}{2} < b_1 + 2b_2 < 3 \cdot \frac{2}{3}$ and $4 \cdot \frac{1}{2} < 2b_3 + 2b_4 < 4 \cdot \frac{2}{3}$, so the left side of the equation is in the interval $(1/2, 1)$ mod 1, and the right side is in the interval $(0, 1/3)$ mod 1. Now, if the members of A were uniformly distributed over the interval $[0, 1]$ modulo 1, then we would expect that approximately $1/6$ of the elements of A fell in the range $(1/2, 2/3)$ modulo 1. This isn't true yet, but consider what happens when we first multiply all the numbers of A by a number r chosen uniformly from some interval $(0, T]$, and then select the subset B of rA that lies in the interval $(1/2, 2/3)$ modulo 1. We now have a subset B such that no four elements of B satisfy

$$rb_1 + 2rb_2 = 2rb_3 + 2rb_4.$$

For a particular element a_i of A, ra_i is distributed over the interval $[0, a_i T]$. If $n < a_i T i \leq n+1$ for an integer n, the probability that ra_i falls in $(1/2, 2/3)$ modulo 1 is at least $\frac{n}{6(n+1)}$ since there are at least n subintervals of the form $(i+1/2, i+2/3)$ in the range $[0, b_i T]$, and the measure of the whole range is $a_i T \leq n+1$. Choosing $T > 6/\min\{a \in A\}$ and using linearity of expectation we find that there is some subset B of cardinality at least $|A|/7$ satisfying the condition.

Problem 2-7 Let F be a family of subsets of $N = \{1, 2, \ldots, n\}$ and suppose $A \not\subset B$ for all $A, B \in \mathcal{F}$. Let $\sigma \in S_n$ be a random permutation of N, and consider the random variable $X = |\{i : \{\sigma(1), \sigma(2) \ldots \sigma(i)\} \in \mathcal{F}\}|$. Prove that $\{\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$.

First, $EX \leq 1$ since $X \leq 1$ for all permutations σ, as otherwise there would be two subsets of $\mathcal{F}$, $\{\sigma(1), \sigma(2), \ldots, \sigma(i)\}$ and $\{\sigma(1), \sigma(2), \ldots, \sigma(j)\}$, one a subset of the other. Next, if we let X_A be the variable X as defined above for the set of subsets $\{A\}$ containning just one element, we see that $EX_A = \binom{n}{|A|}^{-1}$ and $EX = \sum_{A \in \mathcal{F}} EX_A \leq 1$. Using linearity of expectation together with the observation that $\binom{n}{k} \leq \binom{n}{\lfloor n/2 \rfloor}$, we get

$$|\mathcal{F}| \binom{n}{\lfloor n/2 \rfloor}^{-1} \leq EX \leq 1$$

yielding the theorem.