1: The density matrix is
\[ |\psi\rangle\langle\psi| = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{2}}{4} & 0 & \frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}}{4} & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 \\ \frac{\sqrt{2}}{4} & \frac{1}{4} & 0 & \frac{1}{4} \end{pmatrix}. \]

Taking \( \text{Tr}_A \) gives
\[ \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}}{4} & \frac{1}{4} \end{pmatrix}. \]

Taking \( \text{Tr}_B \) gives
\[ \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}. \]

Looking at \( \text{Tr}_A \), one can see that the eigenvectors of \((1, \pm 1)\), which shows that the eigenvalues are \( \frac{1}{2} \pm \frac{\sqrt{2}}{4} \).

An alternative way of calculating is
\[ |\psi\rangle\langle\psi| = \frac{1}{4} (\sqrt{2} |00\rangle + |01\rangle + |11\rangle)(\sqrt{2} \langle 00 | + \langle 01 | + \langle 11 |) \]

Taking \( \text{Tr}_A \) gives
\[ \frac{1}{4} (\sqrt{2} |0\rangle_B + |1\rangle_B) (\sqrt{2} B \langle 0 | + B \langle 1 |) \cdot A \langle 0 |0 \rangle_A + \frac{1}{4} (|1 \rangle_B \langle 1 |) \cdot A \langle 1 |1 \rangle_A \]

(I’ve left out the terms which vanish because they have \( |0\rangle |1 \rangle \) and \( |1\rangle |0 \rangle \) in them.)

2: When we take the partial trace of \( |\psi\rangle\langle\psi| \) we get
\[ \sum_{i,j} a_i a_j^* |v_i\rangle \langle v_j | \cdot \langle w_j |w_i\rangle. \]

Now, we know that this expression must be equal to
\[ \sum_i \mu_i |v_i\rangle\langle v_i |. \]

However, equating coefficients on \( |v_i\rangle \langle v_j | \), we see that this means that \( \mu_i = a_i a_i^* \) and \( \langle w_j |w_i\rangle = 0 \) if \( i \neq j \), showing that the \( |w_j\rangle \) are an orthonormal basis. (We know they are unit vectors because we constructed them that way: the normalization was absorbed into \( a_i \).)
3: We use the formula $\text{CNOT}_{A,B} \text{CNOT}_{B,A} \text{CNOT}_{A,B} = \text{SWAP}$. If we use the fact that a Toffoli gate is a controlled CNOT and a Fredkin gate is a controlled SWAP, we find that

$$\text{Toffoli}_{1,2,3} \text{Toffoli}_{1,3,2} \text{Toffoli}_{1,2,3} = \text{Fredkin}_{1,2,3}$$

where the indexes tell how the qubits fit into the Fredkin gate. (Note that I have defined my Fredkin gate with the qubits in a different order from Nielsen and Chuang, so for Nielsen and Chuang, you would have Fredkin_{3,1,2} in the above formula.)

You can replace the outer two Toffoli gates with CNOT’s by just checking that they work properly if if qubit 1 is $|0\rangle$ — if qubit 1 is $|1\rangle$, then the behavior is the same as the Toffoli. However, if qubit 1 is $|0\rangle$, the middle Toffoli and the Fredkin gate behave as the identity on the last two qubits, and we need to check that $\text{CNOT}^2 = I$, which is correct.

NC 4.28 I need to draw a picture for this … I’ll put it up later.

NC 4.31. All of these equations are straightforward to obtain by matrix multiplication.

You could save yourself a little work by using

$$C\sigma_y C = i \, C\sigma_x C \cdot C\sigma_z C$$

to obtain 4.33 from 4.32 and 4.34. You could also save yourself a little work by using 4.32, 4.33, 4.34, and 4.38 to obtain 4.35, 4.36, 4.37 and 4.39, respectively, by applying the identities $H\sigma_x H = \sigma_z$ and $HC_{1,2}H = HC_{2,1}H$.

NC 4.34.

Let $|v_+\rangle$ and $|v_-\rangle$ be the $\pm 1$ eigenvectors of $U$. We can solve this by looking at the application of the circuit to the input step by step.

We start in the state $|0\rangle \langle \psi_{\text{in}}|$. When we apply the first Hadamard, we obtain

$$2^{-1/2}(|0\rangle + |1\rangle) \langle \psi_{\text{in}}|.$$

When we apply the gate $U$, we get

$$2^{-1/2} |0\rangle \langle \psi_{\text{in}}| + |1\rangle (\alpha |v_+\rangle - \beta |v_-\rangle)$$

where $\alpha |v_+\rangle + \beta |v_-\rangle = |\psi_{\text{in}}\rangle$ is the decomposition of $|\psi_{\text{in}}\rangle$ into the eigenvectors of $U$.

Now, this can be rewritten as

$$2^{-1/2} \alpha(|0\rangle + |1\rangle) |v_+\rangle + 2^{-1/2} \beta(|0\rangle - |1\rangle) |v_-\rangle$$

The next Hadamard turns this into

$$\alpha |0\rangle |v_+\rangle + \beta |1\rangle |v_-\rangle,$$

and measuring the first qubit leaves the second qubit in an eigenstate of $U$. 

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