18.435/2.111 Homework # 2 Solutions

1: First, notice that $RT = \omega TR$.

The state $|\psi\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle)$ works. We want to say that $R^i T^j \otimes I |\psi\rangle$ is orthogonal to $R^i T^j \otimes I |\psi\rangle$ if $i \neq i'$ or $j \neq j'$. Since $RT$ commute at the cost of a phase, we can do this if we show that $\langle \psi | R^i - i T^j - j' |\psi\rangle = 0$. Let’s take the case of $j \neq j'$ first. Then notice $T \otimes I |\psi\rangle = \frac{1}{\sqrt{3}}(|10\rangle + |21\rangle + |02\rangle)$, which is perpendicular to $|\psi\rangle$ no matter what phases you put on the terms, and that all $R$ does is apply phases to the terms. A similar argument works for for $T^2$.

Now, if we have $j = j'$, we need to show that $\langle \psi | R |\psi\rangle$ and $\langle \psi | R^2 |\psi\rangle$ are 0. This is just because $R |\psi\rangle = \frac{1}{\sqrt{3}}(|00\rangle + \omega |11\rangle + \omega^2 |22\rangle)$, and taking the inner product gives $\frac{1}{3}(1 + \omega + \omega^2) = 0$. (and similarly for $R^2$).

2: Suppose we let

$$|\psi\rangle = a |00\rangle + b |01\rangle + c |10\rangle + d |11\rangle$$

$$\sigma_x \otimes I |\psi\rangle = c |00\rangle + d |01\rangle + a |10\rangle + b |11\rangle$$

$$\sigma_z \otimes I |\psi\rangle = a |00\rangle + b |01\rangle - c |10\rangle - d |11\rangle$$

$$i\sigma_y \otimes I |\psi\rangle = c |00\rangle + d |01\rangle - a |10\rangle - b |11\rangle$$

and we can calculate from the fact that these are orthonormal that we must have

$$aa^* + bb^* = \frac{1}{2}$$
$$cc^* + dd^* = \frac{1}{2}$$
$$ac^* + bd^* = 0$$
$$ca^* + db^* = 0$$

But this is exactly the condition that the rows of

$$\sqrt{2} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

are orthonormal. However, if the rows of a matrix are orthonormal, the matrix is unitary and the columns are also orthonormal. The columns being orthonormal is easily checked to be the condition that

$$|\psi\rangle, \ I \otimes \sigma_x |\psi\rangle, \ I \otimes \sigma_y |\psi\rangle, \ I \otimes \sigma_z |\psi\rangle$$

are orthogonal.
4a: Let’s consider $|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$. The rest of the Bell states can be obtained by applying $\sigma_b$ to $|\psi\rangle$, where $b$ is one of $x, y$ or $z$. Suppose that Alice and Bob apply $H$ to both their qubits of $|\psi\rangle$. We know that $|\psi\rangle$ is invariant if the same Basis transformation is applied to both sides, so we get $|\psi\rangle$. Now, since $H\sigma_z H = \sigma_z$, we have

$$(H \otimes H)(\sigma_x \otimes I) |\psi\rangle = (\sigma_z \otimes I)(H \otimes H) |\psi\rangle$$

so $H \otimes H$ interchanges the Bell states $\sigma_x \otimes I |\psi\rangle$ and $\sigma_z \otimes I |\psi\rangle$. A similar argument shows $H \otimes H$ applies a $-1$ phase to $\sigma_y \otimes I |\psi\rangle$.

4b: The only permutations Alice can perform are those performed by $\sigma_x$, $\sigma_z$ and $\sigma_y$. To see that, first note that her transformation $U$ must take $|0\rangle \rightarrow |0\rangle$ and $|1\rangle \rightarrow |1\rangle$ or take $|0\rangle \rightarrow |1\rangle$ and $|1\rangle \rightarrow |0\rangle$. Otherwise, she would obtain both a $|00\rangle$ and a $|10\rangle$ term when applying $U$ to $|00\rangle + |11\rangle$, and these terms don’t simultaneously appear in any Bell state. By possibly multiplying by $\sigma_x$, we obtain a unitary that is diagonal. Now, by considering what happens when this unitary is applied to a Bell state, we realize that it must either be $\alpha I$ or $\alpha \sigma_z$, where $\alpha$ is an arbitrary complex phase.

4c: When we square

$Q \otimes S + R \otimes S + R \otimes T - Q \otimes T$

we get three kinds of terms. The first are those like $QQ \otimes SS = I$. The second are those like $RR \otimes ST$ which are canceled by a term such as $-QQ \otimes ST$. The third are terms of the form $QR \otimes ST$. There are four of these terms, and these add to $[Q, R] \otimes [S, T]$.

Now, we need to show that

$$\langle \psi \mid (Q \otimes S + R \otimes S + R \otimes T - Q \otimes T)^2 \mid \psi \rangle$$

is at most 8, and taking the square root of this equation gives Tsirelsen’s inequality. (This is because $Q \otimes S$ ... is an observable, and thus is diagonalizable, so

$$\langle \psi \mid (Q \otimes S + R \otimes S + R \otimes T - Q \otimes T) \mid \psi \rangle$$

is at most its largest eigenvalue. )

Thus, we need to show

$$\langle \psi \mid [Q, R] \otimes [S, T] \mid \psi \rangle \leq 4.$$

To do this, we can use the fact that the eigenvalues of a tensor product are the product of the eigenvalues, and so we need to show that $[Q, R]$ has eigenvalues of absolute value at most 2. But since

$$[Q, R] = QR - RQ$$

all we need do is show that $QR$ has eigenvalues at most 1 if $Q$ and $R$ have eigenvalues $\pm 1$. But $Q$ and $R$ have eigenvalues $\pm 1$, so they are both Hermitian and unitary. We then see that $QR$ is unitary (it isn’t necessarily Hermitian), so that its eigenvalues are indeed of the form $e^{i\theta}$, $\theta$ real, and this shows that $QR - RQ$ is Hermitian and has eigenvalues between $-2$ and 2.