1: Consider the state

\[ |\psi\rangle_{AB} = \frac{1}{2}(\sqrt{2}|00\rangle + |01\rangle + |11\rangle). \]

Find \( \rho_B = \text{Tr}_A |\psi\rangle \langle \psi| \) and \( \rho_A = \text{Tr}_B |\psi\rangle \langle \psi| \). Show that these matrices have the same eigenvalues by showing \( \text{Tr} \rho_A^2 = \text{Tr} \rho_B^2 \). What are these eigenvalues?

2: we will now prove the property we found in problem 1 holds in general.

Suppose we have a pure state \( |\psi\rangle \) on a tensor product state space \( A \otimes B \). We know \( \rho = \text{Tr}_B |\psi\rangle \langle \psi| \) is a Hermitian matrix, and thus diagonalizable, so \( \rho = \sum_i \mu_i |v_i\rangle \langle v_i| \) for some orthonormal basis \( |v_i\rangle \). Now, since \( |v_i\rangle \) is a basis, we have that \( \psi \) can be expressed as

\[ |\psi\rangle = \sum a_i |v_i\rangle \otimes |w_i\rangle \]

for some unit vectors \( |w_i\rangle \) in \( B \). Take the partial trace of the above expression for \( |\psi\rangle \langle \psi| \) and use the results to show that the \( |w_i\rangle \) are orthonormal, that is, \( \langle w_j|w_i\rangle = \delta_{i,j} \) where \( \delta_{i,j} \) is the Kronecker \( \delta \) function.

By incorporating the phase of \( a_i \) in \( |w_i\rangle \), we get the Schmidt deomposition for a pure state: every pure state can be written as

\[ |\psi\rangle = \sum \sqrt{\mu_i} |v_i\rangle \otimes |w_i\rangle. \]

3: The Fredkin gate, which operates on three bits (or qubits) is a controlled SWAP. That is, you interchange bits two and three if bit one is 1, and do nothing if bit one is 0. Show how to build a Fredkin gate out of three Toffoli gates. Show how to build a Fredkin gate out of one Toffoli gate and several CNOT gates.

In Nielsen and Chuang, do problems 4.28, 4.31, 4.34, and 4.35. For 4.28, I believe you will have to use CNOT and/or Toffoli gates as well as the C-V and C-V\(^4\) gates mentioned in the problem.