18.435/2.111 POVM Lecture

We have already seen how von Neumann, or projective, measurements work. As we have mentioned, these are not the only kind of measurements. The most general kind of measurements are called POVM's. POVM stands for *positive operator valued measure*. We will not be talking about the most general kind of POVM (in which there is a continuous family of outcomes, and sums are replaced by integrals; this is why they are called *measures*). We will only deal with the case where there are a finite number of possible outcomes, over a finite dimensional quantum space. In this case, the elements are Hermitian matrices (finite dimensional positive operators).

Before we can talk about POVM's, we should probably review the case of projective measurements where some of the projections are on subspaces of dimension higher than 1. We use such measurements implicitly, when we measure some qubits of a quantum computer, and leave other qubits untouched, but it has been a while since we gave a formal mathematical definition (if we did), so I will review this now. Suppose we have a set of projectors onto subspaces Π_1, Π_2, ..., Π_k, with the property that these subapces are orthogonal, so

$$\Pi_i \Pi_j = 0 \quad \text{if} \quad i \neq j$$

and these subspaces span the entire space; that is,

$$\sum_{i=1}^{k} = I.$$

Then there is a projective measurement associated with these subspaces which takes a quantum state $|\,\psi\rangle$ to the state

$$\frac{\Pi_i \,|\,\psi\rangle}{\langle\psi\,|\,\Pi_i\,|\,\psi\rangle^{1/2}}$$

with probability

$$\langle\psi\,|\,\Pi_i\,|\,\psi\rangle\,.$$

That is, it projects the state $|\,\psi\rangle$ onto the i'th subspace with probability proportional to the square of the length of this projection. It should be clear that the case where each subspace has dimension 1 corresponds to measuring with respect to an orthonormal basis, the best known case of quantum measurements.

To motivate POVM's let us consider an example. Suppose we have a qubit, so its state is a unit vector in the space with basis $|\,0\rangle$ and $|\,1\rangle$. We can embed this space in a large space by simply adding a number of extra basis vectors. In coordinate notation, this corresponds to adding extra 0's to the coordinates of each state vector.

Let's add the basis vector $|\,2\rangle$ to a qubit. Now, there are orthonormal basis of this 3-dimensional space where none of the basis vectors lie in the subspace containing the original qubit. What happens when we choose one of these bases for a projective

measurement? What effect does this measurement have on the original qubit? Let's look at the example given by the orthonormal basis:

$$\begin{aligned} \tfrac{\sqrt{2}}{\sqrt{3}}\,|\,0\rangle + \tfrac{1}{\sqrt{3}}\,|\,2\rangle \\ -\tfrac{1}{\sqrt{6}}\,|\,0\rangle + \tfrac{1}{\sqrt{2}}\,|\,1\rangle + \tfrac{1}{\sqrt{3}}\,|\,2\rangle \\ -\tfrac{1}{\sqrt{6}}\,|\,0\rangle - \tfrac{1}{\sqrt{2}}\,|\,1\rangle + \tfrac{1}{\sqrt{3}}\,|\,2\rangle\,. \end{aligned}$$

When we take our vector $|\,\psi\rangle$ and measure it using the above basis, what happens? The probability of the first outcome is

$$\left|(\tfrac{\sqrt{2}}{\sqrt{3}}\,\langle 0\,| + \tfrac{1}{\sqrt{3}}\,\langle 2\,|)\,|\,\psi\rangle\right|^2 = \left|\tfrac{\sqrt{2}}{\sqrt{3}}\langle 0|\psi\rangle\right|^2$$

since $\langle 2\,|$ is orthogonal to $|\,\psi\rangle$. If we define the unnormalized quantum states

$$\begin{aligned} |\,e_1\rangle &= \tfrac{\sqrt{2}}{\sqrt{3}}\,|\,0\rangle \\ |\,e_2\rangle &= -\tfrac{1}{\sqrt{6}}\,|\,0\rangle + \tfrac{1}{\sqrt{2}}\,|\,1\rangle \\ |\,e_3\rangle &= -\tfrac{1}{\sqrt{6}}\,|\,0\rangle - \tfrac{1}{\sqrt{2}}\,|\,1\rangle \end{aligned}$$

we similarly see that the probability of outcome i is

$$|\langle e_i|\psi\rangle|^2\,.$$

Now, suppose we have a number of these unnormalized vectors $|\,e_i\rangle$, and we ask when can the above rule for choosing probabilities of outcomes possibly form a measurement. A necessary condition is that the probabilities add to 1; that is,

$$\sum_{i=1}^{k} |\langle e_i|v\rangle|^2 = 1$$

for all unit vectors $|\,v\rangle$ in our quantum state space. This condition is equivalent to

$$\sum_{i=1}^{k} \langle v|e_i\rangle\langle e_i|v\rangle = 1$$

and moving the sum inside the $\langle v\,|\cdot|\,v\rangle$ we have

$$\langle v\,|\left(\sum_{i=1}^{k} |\,e_i\rangle\langle e_i\,|\right)|\,v\rangle = 1$$

for all unit vectors $|\,v\rangle$. However, any Hermitian matrix M satisfying $\langle v\,|\,M\,|\,v\rangle = 1$ for all unit $|\,v\rangle$ must be the identity matrix (one can easily prove all its eigenvalues are 1). Thus, we have the necessary condition

$$\sum_{i=1}^{k} |\,e_i\rangle\langle e_i\,| = I.$$

It turns out that this is necessary and sufficient for a collection of unnormalized vectors $|\, e_i\rangle$ to be the special kind of POVM all of whose elements are rank 1. We next show that if we have a collection of such $|\, e_i\rangle$, we can achieve the above outcome probabilities by using a projective measurement in a higher dimensional space.

Suppose that we have k unnormalized quantum states $|\, e_i\rangle$ in n dimensions such that

$$\sum_{i=1}^{k} |\, e_i\rangle\langle e_i\,| = I.$$

Let us consider the $k \times n$ matrix M obtained by putting these vectors in the columns of a matrix. The entry $M_{i,j}$ is the i'th coordinate of $|\, e_j\rangle$, or $\langle i|e_j\rangle$. The matrix thus looks like

$$\begin{pmatrix} \langle 1|e_1\rangle & \langle 1|e_2\rangle & \dots & \langle 1|e_k\rangle \\ \langle 2|e_1\rangle & \langle 2|e_2\rangle & \dots & \langle 2|e_k\rangle \\ \dots & \dots & \dots & \dots \\ \langle n|e_1\rangle & \langle n|e_2\rangle & \dots & \langle n|e_k\rangle \end{pmatrix}$$

Now, I'd like to claim that all of the *rows* fo this matrix are orthonormal. Let us consider the inner product of row i and row i'. We have that this is

$$\sum_{j=1}^{k} \langle i|e_J\rangle e_j i$$

However, we can move the sum into the middle, where we obtain the identity matrix, becaus of the condition on the $|\, e_j\rangle$. We thus have that the inner product of rows i and i' is

$$\langle i'\,| \left(\sum_{j=1}^{k} |\, e_j\rangle\langle e_j\,|\,|\, i'\rangle \right) = \langle i'|i\rangle = \delta_{i',i}$$

We now have a set of n orthonormal rows in a k-dimensional space. By using Gram-Schmidt, we can extend these to a set of k orthonormal rows. Since any square matrix whose rows are orthonormal is unitary, and thus has orthonomral columns, the columns of this new $k \times k$ matrix correspond to a projective measurement. If this measurement is restricted to act on the n-dimensional subspace given by the first n basis vectors, this becomes the POVM given by the $|\, e_j\rangle$ that we started with.

Thus, we have discovered that if we start with any projective measurement with rank 1 projectors on a large space, and restrict to a smaller space, it can be extressed as a POVM given by a set of unnormalized vectors $|\, e_i\rangle$ with the condition

$$\sum_{i=1}^{k} |\, e_i\rangle\langle e_i\,| = I.$$

Conversely, any POVM with rank 1 elements $|\, e_i\rangle$ satisfying this condition can be expressed as a von Neumann measurement in a higher dimensional space, of which the original space is a subspace.

Now, we can deal with POVM's with elements greater than rank 1. suppose you have a set of matrices E_i with $\sum_{i=1}^{k} E_i = I$ Then, if we let the probability of outcome i when the measurement is applied to $|\,\psi\rangle$ be $p_i = \langle\psi\,|\,E_i\,|\,\psi\rangle$, then essentially the same calculation as above:

$$\begin{aligned}
\sum_i p_i &= \sum_i \langle\psi\,|\,E_i\,|\,\psi\rangle \\
&= \sum_i \mathrm{Tr}\ \ |\,\psi\rangle\langle\psi\,|\,E_i \\
&= \mathrm{Tr}\ \ |\,psi\rangle\langle psi\,|\sum_i E_i \\
&= \mathrm{Tr}\ \ |\,psi\rangle\langle psi\,| \\
&= 1
\end{aligned}$$

shows that the sum of the probabilities is equal to 1. Thus, a set of matrices E_i is potentially a POVM measurement. Can this measurement be realized? We will show that it is always possible to realize such a POVM as a projective measurement in higher dimensions.

How can we do this? What we will do is refine the measurement corresponding to the set $\{E_i\}$ to a measurement corresponding to $\{|\,e_{ij}\rangle\langle e_{ij}\,|\}$. We then find a projective measurement with rank 1 projectors that corresponds to the $\{|\,e_{ij}\rangle\langle e_{ij}\,|\}$, and unrefine this measurement to get a projective measurement that realizes the POVM with elements E_i on the lower dimensional space. Here, the i'th projector Π_i has the same rank as E_i.

How do we refine E_i. We want to find a set of unnormalized state $|\,e_{ij}\rangle$ such that

$$E_i = \sum_j |\,e_{ij}\rangle\langle e_{ij}\,|$$

We can do this in various ways. The most systematic way might be to diagonalize E_i. We then get eigenvalues and eigenvectors λ_{ij} and $|\,f_{ij}\rangle$, and

$$E_i = \sum_j \lambda_j\,|\,f_{ij}\rangle\langle f_{ij}\,| = \sum_j |\,e_{ij}\rangle\langle e_{ij}\,|$$

where $e_{ij} = \lambda_{ij}^{1/2}\,|\,f_{ij}\rangle$. There are in general many other ways to do this decomposition.

Now, we can lift the POVM $|\,e_{ij}\rangle$ to a von Neumann measurement with projectors $|\,v_{ij}\rangle\langle v_{ij}\,|$, where $|\,v_{ij}\rangle$ forms an orthonormal basis. If we let

$$\Pi_i = \sum |\,v_{ij}\rangle\langle v_{ij}\,|$$

then the $\{\Pi_i\}$ form a projective measurement, and we have that for a $|\,psi\rangle$ in the original subspace[1],

$$\langle\psi\,|\,E_i\,|\,\psi\rangle = \left\langle\tilde{\psi}\,\middle|\,\Pi_i\,\middle|\,\tilde{\psi}\right\rangle$$

[1] We should go back and give this a name

where $\tilde{\psi}$ is the quantum stae ψ embedded in the higher dimensional space.

What is the probability of a given outcome in a POVM if you apply it to a density matrix ρ rather than a quantum state $|\,psi\rangle$. You can get the answer by considering the density matrix as a mixture of pure states:

$$\rho = \sum_j \_j \,|\,\psi_j\rangle\langle\psi_j\,|$$

and the answer (calculation can be added later; it's straightforward) is that the probability of outcome i is Tr ρE_i.

There is one more thing to add: in von Neumann measurements, the state of a vector is projected onto one of the subspaces. What happens to vectors in POVM. I'll add this when I get around to it.