SYNOPSIS: DUAL EQUIVALENCE GRAPHS, RIBBON TABLEAUX AND MACDONALD POLYNOMIALS

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1. Introduction

The primary focus of this dissertation is symmetric function theory. The main objectives are to present a new combinatorial construction which may be used to establish the symmetry and Schur positivity of a function expressed in terms of monomials, and to use this method to find a combinatorial description of the Schur expansion for two important classes of symmetric functions, namely LLT and Macdonald polynomials.

Symmetric function theory plays an important role in many areas of mathematics including algebraic combinatorics, representation theory, Lie groups and Lie algebras, algebraic geometry and the theory of special functions. Multiplicities of irreducible components, dimensions of algebraic varieties, and various other algebraic constructions that require the computation of certain integers may be translated to the computation of the coefficients in the expansion of certain generalizations of the Schur basis. Often the coefficients can be identified as generating functions of tableau-like structures, providing a useful and often insightful *combinatorial* formula.

Since their introduction in 1988, Macdonald polynomials have been intensely studied and have been found to have applications in such areas as representation theory, algebraic geometry, group theory, statistics, and quantum mechanics. Unfortunately, given the indirect definition of these polynomials as the unique functions satisfying certain conditions, most results require difficult technical machinery. Recent work by Haglund, Haiman and Loehr has connected the study of Macdonald polynomials to that of LLT polynomials. Though both Macdonald polynomials and LLT polynomials have been shown to be Schur positive using geometric methods, finding a *combinatorial* proof of positivity remains an important open problem in this area.

2. Background

Let h_λ and s_λ denote the complete homogeneous symmetric functions and the Schur functions, respectively. The *Kostka numbers*, $K_{\lambda\mu}$, give the change of basis coefficients between these bases for the ring of symmetric functions, $h_\mu(x) = \sum_\lambda K_{\lambda\mu} s_\lambda(x)$. Both h_μ and s_λ play an important role in representation theory, appearing as the Frobenius characters of representations of S_n and GL_n, with the Schur functions corresponding to irreducible representations and the Kostka numbers giving the highest weight multiplicities for GL_n modules.

In 1988, Macdonald [12] found a remarkable new basis of symmetric functions in two parameters which specializes to Schur functions, complete homogeneous, elementary and monomial symmetric functions and Hall-Littlewood functions, among others. With an appropriate analog of the Hall inner product, the transformed Macdonald polynomials $\widetilde{H}_\mu(x;q,t)$ are uniquely characterized by certain triangularity and orthogonality conditions, from which their symmetry follows. The *Kostka-Macdonald* polynomials, $\widetilde{K}_{\lambda\mu}(q,t)$, are defined by

$$\widetilde{H}_\mu(x;q,t) = \sum_\lambda \widetilde{K}_{\lambda\mu}(q,t) s_\lambda(x).$$

The Macdonald positivity conjecture states that $\widetilde{K}_{\lambda\mu}(q,t) \in \mathbb{N}[q,t]$. Garsia and Haiman [2] conjectured that the transformed Macdonald polynomials $\widetilde{H}_\mu(x;q,t)$ could be realized as the bigraded characters of the diagonal action of S_n on two sets of variables. By analyzing the algebraic geometry of the Hilbert scheme of n points in the plane, Haiman [8] was able to prove this conjecture and consequently establish

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Date: April 27, 2007.

2000 *Mathematics Subject Classification.* Primary 05E10; Secondary 05A30, 33D52.

$\widetilde{K}_{\lambda\mu}(q,t) \in \mathbb{N}[q,t]$. However, finding a *combinatorial* formula for the Kostka-Macdonald polynomials remains an important open problem.

In 1997, Lascoux, Leclerc and Thibon [10] introduced a family of symmetric functions, called LLT polynomials and denoted $\widetilde{G}^{(k)}_{\mu}(x;q)$, which are q-analogs of products of Schur functions. Using Fock space representations of quantum affine Lie algebras constructed by Kashiwara, Miwa and Stern [9], they proved that $\widetilde{G}^{(k)}_{\mu}(x;q)$ is a symmetric function. The Schur-coefficients, $\widetilde{K}^{(k)}_{\lambda\mu}(q)$, are defined by

$$\widetilde{G}^{(k)}_{\mu}(x;q) = \sum_{\lambda} \widetilde{K}^{(k)}_{\lambda\mu}(q) s_{\lambda}(x).$$

Using Kazhdan-Lusztig theory, Leclerc and Thibon [11] proved $\widetilde{K}^{(k)}_{\lambda\mu}(q) \in \mathbb{N}[q]$ for straight shapes μ. This has recently been extended by Grojnowski and Haiman [4] for skew shapes.

An incomplete combinatorial proof of the positivity of $\widetilde{K}^{(2)}_{\lambda\mu}(q)$ given by Carré and Leclerc [1] was later completed by van Leeuwen [13] using the theory of crystals. The proof is quite involved and relies heavily on special properties of $k = 2$ which fail for $k \geq 3$. Finding a combinatorial formula for $\widetilde{K}^{(k)}_{\lambda\mu}(q)$ remains open for $k > 3$, and the proof for $k = 3$ is one of the main results of this dissertation.

Recently, Haglund [5] conjectured, and then proved with Haiman and Loehr [6], a combinatorial formula for Macdonald polynomials expanded in terms of monomial symmetric functions. While this does not yield a combinatorial formula for the Schur expansion, this formula can be seen as expressing $\widetilde{H}_{\mu}(x;q,t)$ as a positive sum of LLT polynomials. In particular, for k fixed, a combinatorial formula for $\widetilde{K}^{(k)}_{\lambda\mu}(q)$ will give a combinatorial formula for $\widetilde{K}_{\lambda\mu}(q,t)$ when μ is a partition with at most k columns.

3. Dual equivalence graphs

The LLT polynomial $\widetilde{G}^{(k)}_{\mu}(x;q)$ is the q-generating function for a k-tuple of semi-standard Young tableaux weighted by a statistic called *cospin*. Following Haiman's advice, I looked at the relation on tableau known as *dual equivalence* [7] to see if there exists an analog for the objects of LLT polynomials. This idea turned out to be quite fruitful.

Similar to an elementary Knuth equivalence which may be defined in terms of three *adjacent* letters, an elementary dual equivalence is defined in terms of three *consecutive* letters. Define a vertex-signed, edge-colored graph on standard Young tableaux by connecting two tableaux with an i-colored edge whenever there is an elementary dual equivalence for $i-1, i, i+1$ between them, and assigning to every tableau T a signature, denoted $\sigma(T)$, which indicates the descent set of the tableau. This graph is called the *standard dual equivalence graph* and has the property that connected components consist of all tableaux of the same shape. Let $\mathcal{G}_{\lambda}$ denote the connected component of the standard dual equivalence graph corresponding to the shape λ.

By analyzing the structure of these graphs, I developed a local characterization of the standard dual equivalence graph depending only on the signatures of the vertices and colors of the edges. A *dual equivalence graph* is defined to be any vertex-signed, edge-colored graph satisfying these axioms. An *isomorphism* between two vertex-signed, edge-colored graphs is a bijection between vertex sets which preserves the signatures of vertices and color adjacency. To justify this axiomatization, I proved the following:

Theorem 1. *Every connected component of a dual equivalence graph is isomorphic to $\mathcal{G}_{\lambda}$ for a unique partition λ.*

Using Gessel's quasi-symmetric functions [3], we may define the generating function for a graph $\mathcal{G}$ with vertex signature function σ to be

$$G(x;q) = \sum_{v \in V(\mathcal{G})} q^{\alpha(v)} Q_{n,\sigma(v)}(x),$$

where α is any statistic on the vertices. For $\mathcal{G}_{\lambda}$ with $\alpha \equiv 0$, this function reduces to s_{λ} by Gessel's formula $s_{\lambda}(x) = \sum_{T \in \mathrm{SYT}(\lambda)} Q_{n,\sigma(T)}(x)$. Therefore Theorem 1 shows that the generating function for a dual equivalence graph is symmetric, and indeed Schur positive, whenever α is constant on connected components of the graph.

Corollary 2. *Let $\mathcal{G}$ be a dual equivalence graph with a vertex statistic α which is constant on connected components. Then*

$$G(x;q) = \sum_{\lambda} \left(\sum_{\mathcal{C} \cong \mathcal{G}_{\lambda}} q^{\alpha(\mathcal{C})} \right) s_{\lambda}(x),$$

where $\mathcal{C}$ ranges over all connected components of $\mathcal{G}$.

4. LLT and Macdonald polynomials

With this theory in place, I realized $\widetilde{G}_{\mu}^{(k)}$ as the generating function for a graph whose vertices are k-tuples of standard Young tableaux. Modifying the rule for an elementary dual equivalence slightly yields a dual equivalence graph for $k = 2$, giving a new proof of positivity for $\widetilde{K}_{\lambda\mu}^{(2)}(q)$ which is far shorter and less complicated than van Leeuwen's proof. I extended this definition to give a graph $\mathcal{G}_{\mu}^{(k)}$ for arbitrary integer k and (skew) shape μ, and conjectured the following.

Conjecture 3. *The vertex-signed, edge-colored graph $\mathcal{G}_{\mu}^{(k)}$ is a dual equivalence graph for which cospin is constant on connected components.*

In this thesis, I prove the conjecture for $k \leq 3$. From this we obtain a combinatorial formula for the Schur expansion of LLT polynomials with $k \leq 3$ by Corollary 2. In light of Haglund's formula, this gives a combinatorial formula for $\widetilde{K}_{\lambda\mu}(q,t)$ when μ is a partition with at most 3 columns. In particular, we have a combinatorial proof of the following.

Theorem 4. *For μ a partition with at most 3 columns, $\widetilde{K}_{\lambda\mu}(q,t) \in \mathbb{N}[q,t]$.*

Though it remains a conjecture that $\mathcal{G}_{\mu}^{(k)}$ is a dual equivalence graph for $k > 3$, much progress in this direction has been made. Many of the steps in the proof that $\mathcal{G}_{\mu}^{(3)}$ is a dual equivalence graph carry over to arbitrary k with no adjustments, though a few of the supporting propositions must be generalized. I am optimistic that the appropriate generalizations will soon be found, and the resulting combinatorial formulae extended to all k.

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